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All tree-level amplitudes in $\mathcal{N} = 4$ SYM

J.M. Drummond and J.M. Henn

LAPTH,[1] Université de Savoie, CNRS,
B.P. 110, F-74941 Annecy-le-Vieux Cedex, France

E-mail: drummond@lapp.in2p3.fr, henn@physik.hu-berlin.de

ABSTRACT: We give an explicit formula for all tree amplitudes in $\mathcal{N} = 4$ SYM, derived by solving the recently presented supersymmetric tree-level recursion relations. The result is given in a compact, manifestly supersymmetric form and we show how to extract from it all possible component amplitudes for an arbitrary number of external particles and any arrangement of external particles and helicities. We focus particularly on extracting gluon amplitudes which are valid for any gauge theory. The formula for all tree-level amplitudes is given in terms of nested sums of dual superconformal invariants and it therefore manifestly respects both conventional and dual superconformal symmetry.

KEYWORDS: AdS-CFT Correspondence, Superspaces, QCD, Supersymmetric gauge theory

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[1]Laboratoire d'Annecy-le-Vieux de Physique Théorique, UMR 5108.

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Contents

1 Introduction

Gluon scattering amplitudes are known to have many remarkable properties. In a recent paper [1], it was discovered that in $\mathcal{N} = 4$ SYM, scattering amplitudes exhibit a new, dual superconformal symmetry. This new symmetry appears in addition to all previously known symmetries of the amplitudes. It was also shown that this dual superconformal symmetry can be understood through the AdS/CFT correspondence, where it appears as a symmetry of the $AdS_5 \times S^5$ string sigma model [2, 3]. In this paper we will construct a solution for all tree-level amplitudes in $\mathcal{N} = 4$ SYM and show explicitly how it respects dual superconformal symmetry.

The first hint at an unexpected simplicity in gluon scattering amplitudes was the formula for the MHV amplitudes conjectured by Parke and Taylor [4] (and later proved by Berends and Giele [5]). For amplitudes having generic helicity configurations, Witten argued that they have remarkable properties in twistor space [6]. This conjecture was verified for NMHV amplitudes [7, 8], however the explicit formulae [9] for these amplitudes are rather complicated. Since tree level gluon amplitudes in $\mathcal{N} = 4$ SYM are equal to gluon amplitudes in any gauge theory, including QCD, it is no restriction to consider amplitudes in $\mathcal{N} = 4$ SYM instead. Keeping this in mind and having observed that $\mathcal{N} = 4$ SYM amplitudes have an additional symmetry, dual superconformal symmetry, it seems natural to write the amplitudes in a manifestly supersymmetric way. The appropriate on-shell $\mathcal{N} = 4$ superspace was introduced by Nair [10], who used it to write down the MHV super-amplitudes. This superspace was employed in [6] to describe amplitudes in super-twistor space and in [11] to express NMHV amplitudes using a supersymmetric version of the CSW rules [12]. Employing this superspace will allow us to make the additional symmetry properties of the amplitudes manifest and hopefully lead to simpler expressions than the previously available ones. Indeed, it was conjectured [1] and later proved [13] that NMHV tree level amplitudes written in this superspace have a remarkably simple form, they are just given by a sum over certain dual superconformal invariants. It seems natural to expect that one can go beyond NMHV amplitudes and that generic N^{p}MHV amplitudes will have a relatively simple form when written in superspace. Since these super-amplitudes are not yet known we compute them in this paper.

The state-of-the-art method for computing tree-level scattering amplitudes in gauge theory are the BCF/BCFW on-shell recursion relations [14, 15]. Recently, these recursion relations have been written for $\mathcal{N} = 4$ SYM in on-shell superspace [16–20]. We will use the form presented in [17–19]. This is precisely the tool we need to study tree-level super-amplitudes for arbitrary helicity configurations. The supersymmetric recursion relations have been used very recently to verify that tree-level scattering amplitudes in $\mathcal{N} = 4$ SYM are covariant under dual superconformal transformations [18].

In this paper, we use the supersymmetric recursion relations to compute tree-level amplitudes in $\mathcal{N} = 4$ SYM. As we will see, writing the recursion relations in superspace makes it significantly simpler to solve them. We use the explicit solutions for NMHV, NNMHV, and NNNMHV amplitudes as examples to study the general pattern and then we present a solution for all amplitudes in terms of nested sums. Our result on NMHV amplitudes confirms the result of [13], while our results for generic non-MHV amplitudes are new.

We then study the symmetries of our solution and show how the conventional superconformal symmetry of $\mathcal{N} = 4$ SYM is realised on the amplitudes. We also study the dual superconformal symmetry that the tree-level super-amplitudes should exhibit [1]. This symmetry is a generalisation of dual conformal symmetry, which first appeared as a property of loop integrals in the perturbative expansion of MHV amplitudes [21–23] and then, in the context of the AdS/CFT correspondence, as the isometry of a T-dual AdS$_5$ in [24, 25] and finally as an anomalous Ward identity for MHV amplitudes [26, 27]. This last manifestation of dual conformal symmetry is based on a conjectured duality between MHV amplitudes and Wilson loops [24, 28, 29] which has been confirmed in perturbation theory up to two loops [26, 27, 30–32]. A review of these developments is given in [33].

The paper is organised as follows. In section 2 we introduce the necessary superspace definitions and briefly review the extension of the BCF recursion relations to superspace. In section 3, we show how to solve the supersymmetric recursion relations in the NMHV case, and in section 4 in the NNMHV case. Based on the previous sections, we give in section 5 the solution to the supersymmetric relations for the generic non-MHV case. In section 6 we discuss both the conventional and dual superconformal symmetry of our solutions. Section 7 serves to explain how to extract gluon scattering amplitudes from our super-amplitudes. Section 8 contains our conclusions. There are two appendices. In appendix A we discuss the behaviour of our results under the collinear limit. In appendix B we give the generators of the ordinary as well as the dual superconformal algebra.

2 Amplitudes and supersymmetric recursion relations

In this paper, we will be discussing colour-ordered scattering amplitudes. The tree-level MHV gluon amplitudes mentioned in the introduction are given by [4, 5][1]

$$A(1^-, 2^+, \ldots, j^-, \ldots, n^+) = \delta^{(4)}(p)\, \frac{\langle 1 j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \ldots \langle n\, 1\rangle}\,, \tag{2.1}$$

where $p = \sum_{i=1}^n \lambda_i^\alpha\, \tilde\lambda_i^{\dot\alpha}$ is the total momentum and $\langle ij\rangle = \lambda_i^\alpha \lambda_{j\,\alpha}$. In order to shed more light on gluon scattering amplitudes of arbitrary helicity configurations and make their symmetries manifest, it is useful to consider scattering amplitudes in $\mathcal{N} = 4$ SYM, which has many exceptional properties. Using Grassmann variables η^A we can write down a super-wavefunction

$$\begin{aligned}\Phi(p, \eta) = {} & G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar\Gamma^D(p) \\ & + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D \epsilon_{ABCD} G^-(p)\,,\end{aligned} \tag{2.2}$$

which incorporates as its components all on-shell states of $\mathcal{N} = 4$ SYM. Since the $\mathcal{N} = 4$ supermultiplet is PCT self-conjugate, we could equally well have chosen an anti-chiral representation (see [1, 13] for more explanations). Then we can define super-amplitudes as

$$\mathcal{A}_n(\lambda, \tilde\lambda, \eta) = \mathcal{A}\left(\Phi_1 \ldots \Phi_n\right)\,. \tag{2.3}$$

In this paper we will be discussing exclusively tree-level amplitudes. The $\mathcal{N} = 4$ supersymmetric version of the MHV tree-level amplitude (2.1) then reads [10]

$$\mathcal{A}_n^{\rm MHV}(\lambda, \tilde\lambda, \eta) = \frac{\delta^{(4)}(p)\, \delta^{(8)}(q)}{\langle 1\, 2\rangle\langle 2\, 3\rangle \ldots \langle n\, 1\rangle}\,, \tag{2.4}$$

where $q = \sum_{i=1}^n \lambda_i^\alpha\, \eta_i^A$. The appearance of $\delta^{(8)}(q)$ is dictated by $\mathcal{N} = 4$ supersymmetry, and can be thought of as imposing super-momentum conservation, just as $\delta^{(4)}(p)$ ensures momentum conservation.

The full tree-level super-amplitude (2.3) contains not just MHV but all possible N^{p}MHV super-amplitudes and has the factors $\delta^{(4)}(p)$ and $\delta^{(8)}(q)$ for the same reason. It is convenient to factor out the MHV tree-level super-amplitude (2.4) and write the remaining factor as $\mathcal{P}_n$,

$$\mathcal{A}_n = \mathcal{A}_n^{\rm MHV} \mathcal{P}_n. \tag{2.5}$$

The factor $\mathcal{P}_n$ has an expansion in the Grassmann parameters η,

$$\mathcal{P}_n = \mathcal{P}_n^{\rm MHV} + \mathcal{P}_n^{\rm NMHV} + \cdots \mathcal{P}_n^{\overline{\rm MHV}}. \tag{2.6}$$

[1] In this paper we omit the standard factor of $i(2\pi)^4$ in the normalisation of the amplitudes.

Of course $\mathcal{P}_n^{\rm MHV} = 1$ while $\mathcal{P}_n^{\rm NMHV}$ has Grassmann degree 4 and the remaining terms increase in Grassmann degree in units of 4 up to $\mathcal{P}_n^{\overline{\rm MHV}}$ which is of degree $4n-16$.

The super-amplitude $\mathcal{A}_n^{\rm MHV}$ contains the pure gluon amplitude (2.1) as a component in the expansion in the Grassmann parameters η_i,

$$\mathcal{A}_n^{\rm MHV} = (\eta_1)^4\,(\eta_j)^4\,A(1^-,2^+,\ldots,j^-,\ldots,n^+) + \cdots\,, \tag{2.7}$$

where $(\eta)^4 = (1/4!)\epsilon_{ABCD}\eta^A\eta^B\eta^C\eta^D$. The full super-amplitude $\mathcal{A}_n$ contains all gluon amplitudes (with arbitrary total helicity) as well as all amplitudes with fermions and scalars in $\mathcal{N}=4$ SYM. The superspace formulation of the amplitudes has the advantage that supersymmetric Ward identities are automatically satisfied. Another advantage is that, as was conjectured in [1] and proved in [13], NMHV amplitudes have a particularly simple form when written in superspace, namely

$$\mathcal{A}_n^{\rm NMHV} = \mathcal{A}_n^{\rm MHV}\mathcal{P}_n^{\rm NMHV} = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}\sum_{1<s<t<n} R_{n;st}\,, \tag{2.8}$$

where $R_{n;st}$ are dual superconformal invariants whose precise form is given in [1] and will be given again shortly.

Let us now quickly introduce the necessary information on the BCF on-shell recursion relations. They express n-point scattering amplitudes in terms of a sum over a product of scattering amplitudes of fewer points [14, 15]. Schematically, they read

$$\mathcal{A} = \sum_{P_i}\sum_h \mathcal{A}_L^h(z_{P_i})\frac{1}{P_i^2}\mathcal{A}_R^{-h}(z_{P_i})\,. \tag{2.9}$$

In (2.9), z_P indicates that in the amplitudes on the r.h.s certain momenta were shifted. The shift can be chosen in many ways. For our purposes it is convenient to shift two adjacent legs according to

$$\hat{\tilde\lambda}_n = \tilde\lambda_n + z_{P_i}\tilde\lambda_1\,, \qquad \hat\lambda_1 = \lambda_1 - z_{P_i}\lambda_n\,. \tag{2.10}$$

Hatted quantities denote the shifted variables. This shift, called an $|n1\rangle$ shift, is depicted in figure 1. Note that the amplitudes $\mathcal{A}_L^h(z_{P_i}), \mathcal{A}_R^{-h}(z_{P_i})$ are on-shell. Indeed, the shift parameter z_P must be chosen such that this is the case, which amounts to saying that the shifted intermediate momentum $\hat P_i = -(\hat\lambda_1\tilde\lambda_1 + \sum_{j=2}^{i-1}\lambda_j\tilde\lambda_j)$ is on-shell, i.e.

$$(\hat P_i)^2 = \left(-\sum_{j=1}^{i-1}\lambda_j\tilde\lambda_j + z_{P_i}\lambda_n\tilde\lambda_1\right)^2 = 0\,. \tag{2.11}$$

Note also that the propagator $1/P_i^2$ in (2.9) is evaluated for unshifted kinematics.

We will use the supersymmetric version of the BCF recursion relations of [17–19]. This amounts to replacing the sum over intermediate states by a superspace integral, and the on-shell amplitudes by super-amplitudes, i.e.

$$\mathcal{A} = \sum_{P_i}\int d^4\eta_{P_i}\mathcal{A}_L(z_{P_i})\frac{1}{P_i^2}\mathcal{A}_R(z_P)\,. \tag{2.12}$$

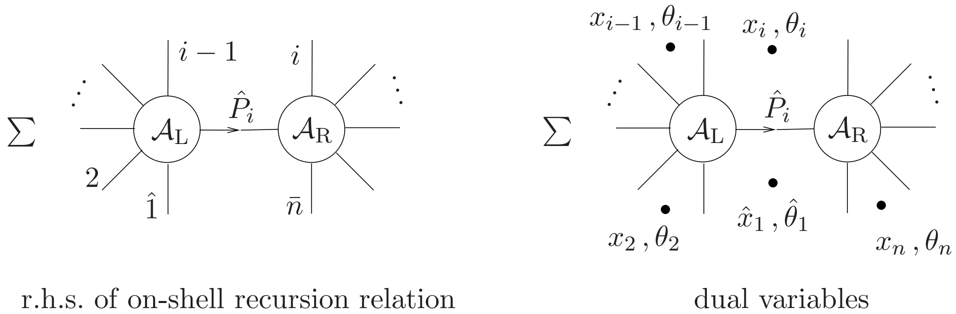

Figure 1. Illustration of the r.h.s of the on-shell recursion relations (2.9), (2.12). The picture on the right illustrates the transition to dual variables.

The validity of the supersymmetric equations can be justified by relating the $z \to \infty$ behaviour of the shifted super-amplitudes $\mathcal{A}(z)$ to the known behaviour of component amplitudes [15] using supersymmetry [17–19].

For the supersymmetric equations, supersymmetry requires that in addition to (2.10) we also have

$$\hat{\eta}_n = \eta_n + z_{P_i}\eta_1 \,. \tag{2.13}$$

In the following sections it will be very useful to use the dual variables [21]

$$\lambda_i\tilde{\lambda}_i = x_i - x_{i+1} \,. \tag{2.14}$$

As was already mentioned, these have a natural generalisation to dual superspace [1], i.e.

$$\lambda_i\eta_i = \theta_i - \theta_{i+1} \,. \tag{2.15}$$

Following [18], in the supersymmetric recursion relations only the following dual variables get shifted,

$$\hat{x}_1 = x_1 - z_{P_i}\lambda_n\tilde{\lambda}_1 \,, \qquad \hat{\theta}_1 = \theta_1 - z_{P_i}\lambda_n\eta_1 \,. \tag{2.16}$$

See figure 1. The fact that all other dual variables remain inert under the shift will prove useful when solving the supersymmetric recursion relations.

3 NMHV tree amplitudes

Here we show that it is straightforward to obtain all NMHV tree amplitudes from the supersymmetric recursion relation (2.12) and knowing the MHV super-amplitudes.

Apart from the n-point MHV super-amplitude (2.4) we need the 3-point $\overline{\text{MHV}}$ amplitude, which can be readily obtained from (2.4) for $n = 3$ by a Grassmann Fourier transform and complex conjugation,

$$\mathcal{A}_3^{\overline{\text{MHV}}}(\lambda,\tilde{\lambda},\eta) = \delta^{(4)}(p)\frac{\delta^{(4)}(\eta_1[23]+\eta_2[31]+\eta_3[12])}{[12][23][31]} \,. \tag{3.1}$$

The form of the three-point $\overline{\text{MHV}}$ amplitude has appeared already in [13, 17–19]. NMHV super-amplitudes have Grassmann degree 12. Looking at (2.12) we see that there is a

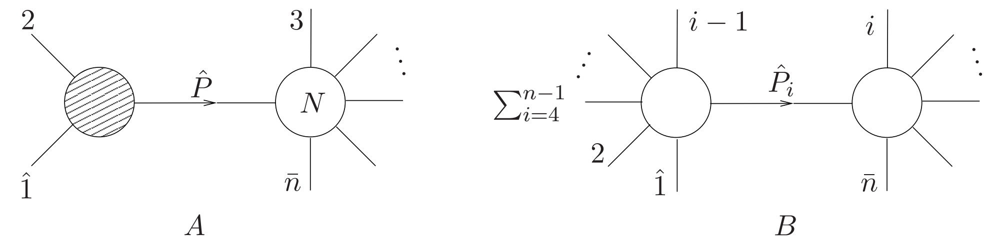

Figure 2. The two contributions to the supersymmetric recursion relation for NMHV amplitudes. We call term B inhomogeneous and A homogeneous. B can be easily computed since it is built from MHV amplitudes only. $\hat{1}$ means that λ_1 is shifted, and $\bar{n}$ means that $\tilde{\lambda}_n$ is shifted.

Grassmann integration, which means that the Grassmann degree of the amplitudes on the r.h.s. of (2.12) must add up to 16. This is only possible in two ways, $4+12$ and $8+8$, which corresponds to taking $\overline{\text{MHV}}_3 + \text{NMHV}$ and MHV + MHV amplitudes for $\mathcal{A}_L, \mathcal{A}_R$, respectively. It is convenient to choose a shift of two neighbouring points, e.g. a $[n1\rangle$ shift. Then the supersymmetric recursion relation for $\mathcal{A}_n^{\text{NMHV}}$ reads

$$\begin{aligned} \mathcal{A}_n^{\text{NMHV}} = &\int \frac{d^4 P}{P^2} \int d^4 \eta_{\hat{P}} \, \mathcal{A}_3^{\overline{\text{MHV}}}(z_P) \mathcal{A}_{n-1}^{\text{NMHV}}(z_P) \\ &+ \sum_{i=4}^{n-1} \int \frac{d^4 P_i}{P_i^2} \int d^4 \eta_{\hat{P}_i} \mathcal{A}_i^{\text{MHV}}(z_{P_i}) \mathcal{A}_{n-i+2}^{\text{MHV}}(z_{P_i}) \\ \equiv &A + B \, . \end{aligned} \tag{3.2}$$

The two terms in (3.2) are depicted in figure 2.

Note that the shifted lines must be on opposite sides of the exchanged line. Note also that the leg n with the anti-holomorphic shift cannot connect to the $\overline{\text{MHV}}_3$ amplitude since this would not be allowed by the kinematics. Similarly, an MHV_i amplitude containing the leg 1 with the holomorphic shift must have at least four legs, which explains the range of i in (3.2).

3.1 Inhomogeneous term

The inhomogeneous term in the recursion relation (3.2) for NMHV amplitudes (corresponding to figure 2 B) can be readily calculated since it is built entirely from the known MHV amplitudes, see (2.4).

By writing, for example, the Grassmann delta function coming from $\mathcal{A}_i^{\text{MHV}}(z_P)$ in the following way,

$$\delta^{(8)}\left(\hat{\lambda}_1\eta_1 + \sum_{j=2}^{i-1}\lambda_j\eta_j - \lambda_{\hat{P}_i}\eta_{\hat{P}_i}\right) = \langle \hat{1}\hat{P}_i\rangle^4 \delta^{(4)}\left(\sum_{j=2}^{i-1}\frac{\langle \hat{1}j\rangle}{\langle \hat{1}\hat{P}_i\rangle}\eta_j - \eta_{\hat{P}_i}\right)\delta^{(4)}\left(\eta_1 + \sum_{j=2}^{i-1}\frac{\langle j\hat{P}_i\rangle}{\langle \hat{1}\hat{P}_i\rangle}\eta_j\right), \tag{3.3}$$

the integration over $\eta_{\hat{P}_i}$ can be carried out straightforwardly. In this way, we obtain the following contribution to the n-point NMHV amplitude:

$$B = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\prod_{j=1}^{n}\langle j\, j+1\rangle} \sum_{i=4}^{n-1} R_{n;2\,i} \, . \tag{3.4}$$

Here $R_{r;st}$ is a dual superconformal invariant introduced in [1],

$$R_{r;st} = \frac{\langle s\ s-1\rangle\langle t\ t-1\rangle\delta^{(4)}(\Xi_{r;st})}{x_{st}^2\langle r|x_{rs}x_{st}|t\rangle\langle r|x_{rs}x_{st}|t-1\rangle\langle r|x_{rt}x_{ts}|s\rangle\langle r|x_{rt}x_{ts}|s-1\rangle}\,. \tag{3.5}$$

The Grassmann odd quantity $\Xi_{r;st}$ is given by

$$\Xi_{r;st} = \langle r|x_{rs}x_{st}|\theta_{tr}\rangle + \langle r|x_{rt}x_{ts}|\theta_{sr}\rangle\,. \tag{3.6}$$

Here we used the dual variables x_i and θ_i defined by (2.14) and (2.15).

In the following we will often deal with the quantity $\Xi_{n;st}$ for $1 < s < t < n$. It is instructive to switch from the dual θ_i in (3.6) to the η_i,

$$\Xi_{n;st} = \langle n|\left[x_{ns}x_{st}\sum_{i=t}^{n-1}|i\rangle\eta_i + x_{nt}x_{ts}\sum_{i=s}^{n-1}|i\rangle\eta_i\right]\,, \tag{3.7}$$

to see that $\Xi_{n;st}$ is independent of η_n and η_1. Alternatively, using the $\delta^{(8)}(q)$ present in all physical amplitudes to rewrite the sums we can obtain

$$\delta^{(8)}(q)\,\Xi_{n;st} = -\delta^{(8)}(q)\,\langle n|\left[x_{ns}x_{st}\sum_{i=1}^{t-1}|i\rangle\eta_i + x_{nt}x_{ts}\sum_{i=1}^{s-1}|i\rangle\eta_i\right]\,, \tag{3.8}$$

such that the only dependence on η_{n-1} and η_n on the l.h.s. of (3.8) is contained in $\delta^{(8)}(q)$. These facts will be useful in the following sections when carrying out superspace integrations.

Moreover, it is useful to realise that terms like $\langle r|x_{rs}x_{st}|t\rangle$ in (3.5) and similar terms in (3.6) can always be written as

$$\langle r|x_{rs}x_{st}|t\rangle = \langle r|x_{r+1\,s}x_{st}|t\rangle\,, \tag{3.9}$$

such that it is clear that they only depend explicitly on λ_r, but not on $\tilde{\lambda}_r$.

3.2 5-point example

In [18], the supersymmetric recursion relations were examined for the example of the five-point $\overline{\text{MHV}}$ amplitude. We will also examine this example here as it is the first example of an NMHV amplitude. For five points, $\text{NMHV}_5 = \overline{\text{MHV}}_5$, and therefore we could have obtained the NMHV_5 amplitude from a Grassmann Fourier transform of the MHV_5 amplitude [13].

We immediately see that only the second term in (3.2) contributes, because there is no four-point amplitude of Grassmann degree 12. Hence for five points, the complete amplitude is given by (3.4), i.e.

$$\mathcal{A}_5^{\text{NMHV}} = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\prod_{j=1}^5\langle j\,j+1\rangle}\,R_{5;2\,4}\,. \tag{3.10}$$

We remark that the invariant $R_{5;2,4}$ can be further simplified, but this is a special feature of the $n = 5$ case.

Another remark is that the super-amplitude must have cyclic symmetry. This allows us to conclude that

$$\delta^{(8)}(q)\, R_{5;2\,4} = \delta^{(8)}(q) R_{1;3\,5} = \delta^{(8)}(q)\, R_{2;4\,1} = \delta^{(8)}(q)\, R_{3;5\,2} = \delta^{(8)}(q)\, R_{4;1\,3}\,. \tag{3.11}$$

This is just the first example of the more general identity for n points, given in [13], where

$$\delta^{(8)}(q) \sum_{s,t} R_{r;st} = \delta^{(8)}(q) \sum_{s,t} R_{r';st}\,, \tag{3.12}$$

where the sum goes over all values of s,t such that r,s,t (or r',s,t) are ordered cyclically with r and s (or r' and s) and s and t separated by at least two.

3.3 General solution for NMHV amplitudes

It can be seen that there is a simple pattern to how the n-point solution is generated from the $(n-1)$-point one. Let us check that the formula

$$\mathcal{A}_n^{\rm NMHV} = \mathcal{A}_n^{\rm MHV} \mathcal{P}_n^{\rm NMHV} = \frac{\delta^{(4)}(p)\, \delta^{(8)}(q)}{\langle 1\,2\rangle\langle 2\,3\rangle \ldots \langle n\,1\rangle} \sum_{2\le s<t\le n-1} R_{n;st}\,, \tag{3.13}$$

indeed solves the supersymmetric recursion relation (3.3). In this formula we are assuming that s and t are separated by at least two. Comparing to (3.10) we see that for $n=5$ the form (3.13) is correct.

We now proceed to prove (3.13) by induction. Let us assume that the form (3.13) is valid for $n-1$ points. Then it follows from the cyclicity of super-amplitudes that (3.12) is also true for $n-1$ points. Now, we notice that $\mathcal{A}_{n-1}^{\rm NMHV}(z_P)$ in the homogeneous term, A on the r.h.s. of (3.2), only involves the quantities $R_{n-1;st}$ where the first subscript is always equal to $n-1$. Cyclic symmetry allows us to insert $\mathcal{A}_{n-1}^{\rm NMHV}(z_P)$ into (3.2) in our favourite orientation. It is convenient to insert it such that the legs $\{1,2,3,\ldots,n-1\}$ of $\mathcal{A}_{n-1}^{\rm NMHV}(z_P)$ are identified with the legs $\{\hat{P},3,4,\ldots,n\}$ in the recursion relation (see figure 2),

$$A = \int \frac{d^4P}{P^2} \int d^4\eta_{\hat{P}}\, \mathcal{A}_3^{\overline{\rm MHV}}(z_P) \mathcal{A}_{n-1}^{\rm MHV} \mathcal{P}_{n-1}(\hat{P},3,\ldots,\bar{n}). \tag{3.14}$$

After carrying out this change of labels in $\mathcal{A}_{n-1}^{\rm NMHV}(z_P)$ is is clear from equations (3.7) and (3.9) that the obtained $R_{n;st}$ does not depend on $\eta_{\hat{P}}$. Indeed the range of η-dependence is only $\{\eta_3,\ldots\eta_{n-1}\}$. When the lower summation variable attains its minimum value, there is an explicit dependence on the spinor $\langle\hat{P}|$. However, due to the three-point kinematics, this spinor is proportional to $\langle 2|$ and since it appears homogeneously in R with degree zero it can simply be replaced by $\langle 2|$. Thus we find

$$A = \frac{\delta^{(4)}(p)\, \delta^{(8)}(q)}{\prod_{j=1}^n \langle j\, j+1\rangle} \sum_{3\le s<t\le n-1} R_{n;st}\,. \tag{3.15}$$

We see that (3.4) is just the missing first term (for $s=2$) to complete (3.15) to the ansatz (3.13) for n points, i.e.

$$A + B = \mathcal{A}_n^{\rm NMHV} = \frac{\delta^{(4)}(p)\, \delta^{(8)}(q)}{\prod_{j=1}^n \langle j\, j+1\rangle} \sum_{2\le s<t\le n-1} R_{n;st}\,. \tag{3.16}$$

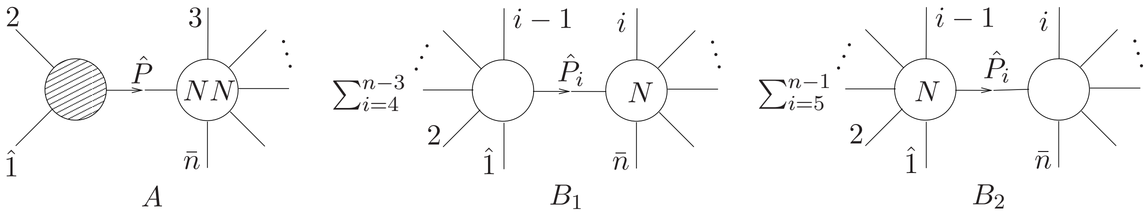

Figure 3. The three contributions to the supersymmetric recursion relation for NNMHV amplitudes.

This completes the inductive proof. Cyclicity of the super-amplitude justifies the general identity (3.12). To prepare for the notation that we use in section 5, we will rewrite the formula for NMHV amplitudes with different labels and using $\mathcal{P}_n^{\rm NMHV}$ instead of $\mathcal{A}_n^{\rm NMHV} = \mathcal{A}_n^{\rm MHV}\mathcal{P}_n^{\rm NMHV}$,

$$\mathcal{P}_n^{\rm NMHV} = \sum_{2\leq a_1,b_1\leq n-1} R_{n;a_1b_1}\,. \tag{3.17}$$

The reason is that in the following sections we will derive a formula for the full $\mathcal{P}_n$ defined in (2.5) and we will encounter generalisations of the invariant $R_{n;a_1b_1}$ with multiple labels.

Thus we see that the result (3.13) which was conjectured in [1] and derived in [13] follows very naturally from the recursion relations. Of course it should be equivalent to the result found in [11] using a supersymmetrised version of the CSW rules [12].

4 NNMHV tree amplitudes

Before we generalise to all tree-level super-amplitudes, it is useful to look first at the next case, namely NNMHV amplitudes. In examining the recursion relation in this case we will find new features which will help us find the solution for the full super-amplitude in the next section.

The recursive relation for NNMHV amplitudes reads

$$\begin{aligned}\mathcal{A}_n^{\rm NNMHV} &= \int \frac{d^4P}{P^2}\int d^4\eta_{\hat P}\,\mathcal{A}_3^{\overline{\rm MHV}}(z_P)\mathcal{A}_{n-1}^{\rm NNMHV}(z_P)\\ &\quad+\sum_{i=4}^{n-3}\int\frac{d^4P_i}{P_i^2}\int d^4\eta_{\hat P_i}\mathcal{A}_i^{\rm MHV}(z_{P_i})\mathcal{A}_{n-i+2}^{\rm NMHV}(z_{P_i})\\ &\quad+\sum_{i=5}^{n-1}\int\frac{d^4P_i}{P_i^2}\int d^4\eta_{\hat P_i}\mathcal{A}_i^{\rm NMHV}(z_{P_i})\mathcal{A}_{n-i+2}^{\rm MHV}(z_{P_i}) \equiv A+B_1+B_2\,.\end{aligned} \tag{4.1}$$

It is very similar to the recursion relation for NMHV amplitudes, and as we will show presently, it can be solved in a similarly straightforward manner.

Before we derive the solution to (4.1), it is helpful to introduce some new notation. Firstly we will introduce generalisations of the R-invariant which we used to express the NMHV amplitudes. The new quantities have many pairs of labels and are given by

$$R_{n;b_1a_1;b_2a_2;\ldots;b_ra_r;ab} = \frac{\langle a\ a-1\rangle\langle b\ b-1\rangle\delta^{(4)}(\langle\xi|x_{a_ra}x_{ab}|\theta_{ba_r}\rangle + \langle\xi|x_{a_rb}x_{ba}|\theta_{aa_r}\rangle)}{x_{ab}^2\langle\xi|x_{a_ra}x_{ab}|b\rangle\langle\xi|x_{a_ra}x_{ab}|b-1\rangle\langle\xi|x_{a_rb}x_{ba}|a\rangle\langle\xi|x_{a_rb}x_{ba}|a-1\rangle}\,, \tag{4.2}$$

where the chiral spinor $\langle\xi|$ is given by

$$\langle\xi| = \langle n|x_{nb_1}x_{b_1a_1}x_{a_1b_2}x_{b_2a_2}\dots x_{b_ra_r}\,. \tag{4.3}$$

In the case where there is only one pair of labels ab after the initial label n, (4.2) is just the R-invariant (3.5) we have already seen appearing in the NMHV amplitudes. The cases where there is more than one pair are generalisations. The new quantities $R_{n;b_1a_1;\dots;b_ra_r;ab}$ are invariant under dual conformal symmetry, but not (except for the case $R_{n;ab}$) under dual superconformal symmetry. However they will always appear in the amplitude together with additional factors which will combine with them to make dual superconformal invariants. We will explore this point in more detail in section 6.

We also need to introduce a second piece of notation. Just as we have already seen in the NMHV case, the R-invariants will always appear in the amplitude with a summation over the last pair of labels (the summation will always take place over the region where a and b are separated by at least two, $a < b-1$), i.e. in the form,

$$\sum_{L\leq a<b\leq U} R_{n;b_1a_1;\dots;b_ra_r;ab}\,. \tag{4.4}$$

We will write superscripts on the R-invariants to indicate special behaviour for the boundary terms when $a = L$ or $b = U$. Specifically we write

$$\sum_{L\leq a<b\leq U} R^{l_1\dots l_p;u_1\dots u_q}_{n;b_1a_1;\dots;b_ra_r;ab}\,. \tag{4.5}$$

This notation means the following. For the terms in the sum where $a = L$ we replace the explicit dependence on $\langle L-1|$ in (4.2) in the following way,

$$\langle L-1| \longrightarrow \langle n|x_{nl_1}x_{l_1l_2}x_{l_2l_3}\dots x_{l_{p-1}l_p}\,. \tag{4.6}$$

Similarly, for the terms in the sum where $b = U$ we replace the explicit dependence on $\langle U|$ in (4.2) in the following way,

$$\langle U| \longrightarrow \langle n|x_{nu_1}x_{u_1u_2}x_{u_2u_3}\dots x_{u_{q-1}u_q}\,. \tag{4.7}$$

Of course there is one term in the sum where $a = L$ and $b = U$ where both replacements occur. When no replacement is to be made on one of the boundaries we will write the superscript 0.

Using this notation we will now state the result for all NNMHV amplitudes. As usual we have

$$\mathcal{A}_n^{\rm NNMHV} = \mathcal{A}_n^{\rm MHV}\mathcal{P}_n^{\rm NNMHV}. \tag{4.8}$$

Then the factor $\mathcal{P}_n^{\rm NNMHV}$ is given by

$$\mathcal{P}_n^{\rm NNMHV} = \sum_{2\leq a_1,b_1\leq n-1} R^{0;0}_{n;a_1b_1}\left[\sum_{a_1+1\leq a_2,b_2\leq b_1} R^{0;a_1b_1}_{n;b_1a_1;a_2b_2} + \sum_{b_1\leq a_2b_2\leq n-1} R^{a_1b_1;0}_{n;a_2b_2}\right]. \tag{4.9}$$

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The superscripts 0;0 on the outer R-invariant $R^{0;0}_{n;a_1,b_1}$ simply mean that nothing special happens at the boundaries 2 and $n-1$, as is also the case in formula (3.17) for the NMHV amplitudes. Thus this expression differs from the formula for the NMHV amplitudes in that the factor in the square brackets is not 1 but is itself a sum over R-invariants. For the sums of R-invariants in the square brackets the superscripts denote the fact that there are non-trivial boundary effects (at the upper boundary for the first term and the lower boundary for the second).

Let us now demonstrate the validity of formula (4.9). The first step is to calculate the two inhomogeneous terms in the recursion relation, labelled B_1 and B_2 in figure 3. We start with the calculation of B_1 which corresponds to the second term on the r.h.s. of the recursion relation (4.1). This term is very similar to the inhomogeneous term B that we encountered for the NMHV amplitudes in section 3. The difference from that case is that for B_1 we have an additional factor of $\mathcal{P}^{\rm NMHV}$,

$$B_1 = \sum_{i=4}^{n-3} \int \frac{d^4 P_i}{P_i^2} \int d^4\eta_{\hat{P}_i} \mathcal{A}_i^{\rm MHV}(z_{P_i}) \mathcal{A}_{n-i+2}^{\rm MHV}(z_{P_i}) \mathcal{P}_{n-i+2}^{\rm NMHV}(z_{P_i})\,. \tag{4.10}$$

Thanks to the cyclic symmetry of the amplitudes, we have the freedom to insert the NMHV factor in our preferred orientation. We will choose to insert it so that the legs $\{1,2,\ldots,n-i+2\}$ of the subamplitude correspond to the legs $\{\hat{P},i,\ldots,\bar{n}\}$ in the recursion relation, as shown in figure 3. With this choice we find that the R-invariants appearing in the factor of $\mathcal{P}^{\rm NMHV}_{n-i+2}$ (see equations (3.17) and (3.5)) do not depend on $\eta_{\hat{P}}$ and are therefore inert under the Grassmann integral. The integration is therefore identical to that which we performed in the calculation of B in subsection 3.1 and we obtain a result very similar to equation (3.4),

$$B_1 = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\prod_{j=1}^{n}\langle j\, j+1\rangle} \sum_{i=4}^{n-1} R_{n;2\,i} \mathcal{P}^{\rm NMHV}_{n-i+2}(\hat{P},\ldots,\bar{n})\,. \tag{4.11}$$

Now, if we compare the factor $\mathcal{P}^{\rm NMHV}_{n-i+2}(\hat{P},\ldots,\bar{n})$ against the general formula for NMHV amplitudes (3.17) and the definition of the R-invariants (3.5), we see that we can write it as

$$\mathcal{P}^{\rm NMHV}_{n-i+2}(\hat{P},\ldots,\bar{n}) = \sum_{i\le s,t\le \bar{n}-1} R_{\bar{n};st}(\hat{P},\ldots,\bar{n}), \tag{4.12}$$

where the notation indicates that we must remember that legs associated to this factor form the ordered set $\{\hat{P},\ldots,\bar{n}\}$. Thus when $s=i$ the explicit dependence of R on $\langle s-1|$ becomes a dependence on $\langle\hat{P}|$. The spinor $\langle\hat{P}|$ appears once in the numerator and once in the denominator of the relevant R-invariants. For these boundary terms in the sum we will write the dependence on $\hat{P}$ in the following way. First we multiply both the numerator and denominator by $\langle n1\rangle[1\hat{P}]$. Then we can see that for any factor which has the spinor $\langle\hat{P}|$ in it we can write

$$\langle n1\rangle[1\hat{P}]\langle\hat{P}|\ldots = \langle n1\rangle[1|P\ldots = \langle n1\rangle[1|x_{1i}\ldots = \langle n1\rangle[1|x_{2i}\ldots = \langle n|x_{12}x_{2i}\ldots = \langle n|x_{n2}x_{2i}\ldots \tag{4.13}$$

So for the boundary terms $s = i$ we have a modification of the R-invariant where the explicit dependence on the spinor $\langle i-1|$ is replaced in the following way,

$$\langle i-1| \longrightarrow \langle n|x_{n2}x_{2i}\,. \tag{4.14}$$

The remaining terms in the sum (4.12) are just unmodified R-invariants. This is why we introduced the idea of superscripts on the R-invariants. The replacement (4.14) is an example of the lower limit replacement (4.6). The total effect is summarised by the following formula for B_1,

$$B_1 = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\prod_{j=1}^{n}\langle j\,j+1\rangle} \sum_{i=4}^{n-1} R_{n;2\,i} \sum_{i\leq s,t\leq n-1} R^{2i;0}_{n;st}\,. \tag{4.15}$$

Now let us address the second inhomogeneous term B_2. This is similar to the term B_1 which we already calculated, but this time the factor of $\mathcal{P}_i^{\mathrm{NMHV}}$ appears in the left factor instead of the right factor,

$$B_2 = \sum_{i=4}^{n-3} \int \frac{d^4P_i}{P_i^2} \int d^4\eta_{\hat{P}_i} \mathcal{A}_i^{\mathrm{MHV}}(z_{P_i})\mathcal{P}_i^{\mathrm{NMHV}}(z_{P_i})\mathcal{A}_{n-i+2}^{\mathrm{MHV}}(z_{P_i}). \tag{4.16}$$

Again we can choose the legs of the left subamplitude so that the R-invariants contained in $\mathcal{P}_i^{\mathrm{NMHV}}$ are inert under the Grassmann integration. One way to do this is to have the legs $\{1,\ldots,i\}$ match up with legs $\{2,\ldots,-\hat{P},\hat{1}\}$ in the recursion relation. In much the same way as for B_1 this allows us to write

$$B_2 = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\prod_{j=1}^{n}\langle j\,j+1\rangle} \sum_{i=4}^{n-1} R_{n;2\,i} \sum_{3\leq s,t\leq \hat{P}} R_{\hat{1};st}(2,\ldots,-\hat{P},\hat{1})\,, \tag{4.17}$$

where again the notation is to remind us that the legs associated with the R-invariants under the second sum form the ordered set $\{2,\ldots,-\hat{P},\hat{1}\}$. Thus when $t = \hat{P}$ we will have an explicit dependence on the spinor $\hat{P}$ in the R-invariants. Using exactly the same reasoning as in (4.13) above we see that the resulting R-invariants will have the upper boundary replacement,

$$\langle i| \longrightarrow \langle n|x_{n2}x_{2i}\,. \tag{4.18}$$

In addition, there is a new feature in the calculation of B_2. This arises from the fact that the last leg in the subamplitude is $\hat{1}$ and not n. Therefore the spinor $\langle\hat{1}|$ appears four times in the numerator and four times in the denominator of every R-invariant. We can deal with this by writing the explicit expression for $\langle\hat{1}|$,

$$\langle\hat{1}| = \langle 1| - z_{P_i}\langle n| = \langle 1| - \frac{x_{1i}^2}{\langle n|x_{1i}|1]}\langle n| = \frac{\langle n|x_{1i}(x_{12}-x_{1i})}{\langle n|x_{1i}|1]} = \frac{\langle n|x_{ni}x_{i2}}{\langle n|x_{1i}|1]}. \tag{4.19}$$

Since $\langle\hat{1}|$ appears homogeneously in the R-invariants, the denominator $\langle n|x_{1i}|1]$ in (4.19) drops out and we effectively have the following replacement in the R-invariants,

$$\langle n| \longrightarrow \langle n|x_{ni}x_{i2}\,. \tag{4.20}$$

Taking into account both effects (4.18) and (4.20) we find that B_2 is given by the following formula,

$$B_2 = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\prod_{j=1}^{n}\langle j\,j+1\rangle} \sum_{i=4}^{n-1} R_{n;2\,i} \sum_{3\le s,t\le i} R^{0;2i}_{n;i2;st}\,. \tag{4.21}$$

The upper limit replacement (4.18) is responsible for the non-trivial right-superscript, while the extension of the spinor $\langle n|$ in (4.20) is responsible for the fact that we have the first example of the generalised R-invariants, defined in equation (4.2).

Now we are in a position to justify the formula (4.9) for the NNMHV amplitudes. We will proceed by induction and assume that (4.9) is true for $(n-1)$-point amplitudes. Then we can treat the homogeneous term (labelled A in figure 3) in exactly the same way as for NMHV amplitudes. Again we will insert $\mathcal{A}^{\mathrm{NNMHV}}_{n-1}(z_P)$ so that the legs $\{1,\ldots,n-1\}$ of the subamplitude coincide with legs $\{\hat{P},3,\ldots,\bar{n}\}$ of the recursion relation,

$$A = \int \frac{d^4P}{P^2} \int d^4\eta_{\hat{P}}\, \mathcal{A}^{\overline{\mathrm{MHV}}}_3(z_P)\mathcal{A}^{\mathrm{MHV}}_{n-1}(z_P)\mathcal{P}_{n-1}(\hat{P},3,\ldots,\bar{n}). \tag{4.22}$$

With this choice we find all R-invariants are again inert under the Grassmann integral. To see this, we first note that the outer R-invariant in $\mathcal{P}_{n-1}(\hat{P},3,\ldots,\bar{n})$ (see (4.9)) is the same as in the NMHV case. We have already seen in subsection 3.3 that this does not depend on $\eta_{\hat{P}}$ and so is inert under the Grassmann integral. The other R-invariants in $\mathcal{P}_{n-1}(\hat{P},3,\ldots,\bar{n})$ (which come from the terms in square brackets in (4.9)) also do not depend on $\eta_{\hat{P}}$. The first term in the square brackets depends on $\{\eta_3,\ldots,\eta_{n-2}\}$, as can be seen from equations (4.2) and (2.15), while the second depends on $\{\eta_3,\ldots,\eta_{n-1}\}$ just like the outer R-invariant.

Just as for the case of the NMHV amplitudes, when the outermost lower summation variable (which corresponds to a_1 in equation (4.9)) reaches its lowest value, there is an explicit dependence on the spinor $\langle\hat{P}|$. However, as in the NMHV case, this can simply be replaced by $\langle 2|$ due to the three-point kinematics. Thus we obtain the following simple result for A,

$$A = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\prod_{j=1}^{n}\langle j\,j+1\rangle} \sum_{3\le a_1,b_1\le n-1} R^{0;0}_{n;a_1b_1}\left[\sum_{a_1+1\le a_2,b_2\le b_1} R^{0;a_1b_1}_{n;b_1a_1;a_2b_2} + \sum_{b_1\le a_2 b_2\le n-1} R^{a_1b_1;0}_{n;a_2b_2}\right]. \tag{4.23}$$

Combining the results from A, B_1 and B_2 we find

$$\begin{aligned} A+B_1+B_2 &= \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\prod_{j=1}^{n}\langle j\,j+1\rangle} \sum_{2\le a_1,b_1\le n-1} R^{0;0}_{n;a_1b_1}\left[\sum_{a_1+1\le a_2,b_2\le b_1} R^{0;a_1b_1}_{n;b_1a_1;a_2b_2} + \sum_{b_1\le a_2 b_2\le n-1} R^{a_1b_1;0}_{n;a_2b_2}\right] \\ &= \mathcal{A}^{\mathrm{MHV}}_n \mathcal{P}^{\mathrm{NNMHV}}_n. \end{aligned} \tag{4.24}$$

We know formula (4.9) is correct for the six-point amplitudes, since the inhomogeneous terms are the only contributions to this case. Therefore we have completed the inductive justification of the result (4.9) for the NNMHV amplitudes.

5 All tree amplitudes

It is simple to continue the analysis of the preceding sections to N^3MHV, N^4MHV amplitudes and so on. The supersymmetric recursion relation for a generic N^pMHV amplitude reads

$$
\begin{aligned}
\mathcal{A}_n^{\mathrm{N}^p\mathrm{MHV}} &= \int \frac{d^4P}{P^2} \int d^4\eta_{\hat{P}}\, \mathcal{A}_3^{\overline{\mathrm{MHV}}}(z_P)\mathcal{A}_{n-1}^{\mathrm{N}^p\mathrm{MHV}}(z_P) \\
&\quad + \sum_{m=0}^{p-1} \sum_i \int \frac{d^4P_i}{P_i^2} \int d^4\eta_{\hat{P}_i} \mathcal{A}_i^{\mathrm{N}^m\mathrm{MHV}}(z_{P_i})\mathcal{A}_{n-i+2}^{\mathrm{N}^{(p-m-1)}\mathrm{MHV}}(z_{P_i})\,. \qquad (5.1)
\end{aligned}
$$

At each stage one obtains the universal prefactor $\mathcal{A}_n^{\mathrm{MHV}}$ while the R-invariants from the right-hand factor in the second line are left unchanged and those from the left-hand factor acquire an additional extension, just as in the case of the NNMHV amplitudes. As before, one must carefully take into account the behaviour of the boundary terms in the sums. For example, we find that the N^3MHV amplitudes are given by the formula,

$$
\begin{aligned}
\mathcal{P}_n^{\mathrm{N}^3\mathrm{MHV}} = \sum_{2\leq a_1,b_1\leq n-1} R_{n;a_1b_1}\Bigg[& \sum_{a_1+1\leq a_2,b_2\leq b_1} R^{0;a_1b_1}_{n;b_1a_1;a_2b_2}\Bigg(\sum_{a_1+1\leq a_3,b_3\leq b_2} R^{0;b_1a_1a_2b_2}_{n;b_1a_1;b_2a_2;a_3b_3} \\
& \qquad\qquad + \sum_{b_2\leq a_3,b_3\leq b_1} R^{b_1a_1a_2b_2;a_1b_1}_{n;b_1a_1;a_3b_3}\Bigg) \\
& + \sum_{a_1+1\leq a_2,b_2\leq b_1} R^{0;a_1b_1}_{n;b_1a_1;a_2b_2} \sum_{b_1\leq a_3,b_3\leq n-1} R^{a_1b_1;0}_{n;a_3b_3} \\
& + \sum_{b_1\leq a_2,b_2\leq n-1} R^{a_1b_1;0}_{n;a_2b_2}\Bigg(\sum_{a_2+1\leq a_3,b_3\leq b_2} R^{0;a_2b_2}_{n;b_2a_2;a_3b_3} + \sum_{b_2\leq a_3,b_3\leq n-1} R^{a_2b_2;0}_{n;a_3b_3}\Bigg)\Bigg]\,.
\end{aligned}
\qquad (5.2)
$$

If we take the terms in the outermost sum where $a_1 = 2$ then the three lines correspond to the three different inhomogeneous terms in the recursion relation $\mathcal{A}_L^{\mathrm{NNMHV}}\mathcal{A}_R^{\mathrm{MHV}}$, $\mathcal{A}_L^{\mathrm{NMHV}}\mathcal{A}_R^{\mathrm{NMHV}}$ and $\mathcal{A}_L^{\mathrm{MHV}}\mathcal{A}_R^{\mathrm{NNMHV}}$. As before, the superscripts on the R-invariants indicate the lower and upper limit replacements. The formula (5.2) can be justified by induction, just as we saw in the cases of the NMHV and NNMHV amplitudes. We will not give the argument here because in this section we will give an inductive argument which proves a general formula for the whole super-amplitude (i.e. for all N^pMHV amplitudes for all p).

It is helpful to notice that the first and second lines of (5.2) can be combined so that we have

$$
\begin{aligned}
\mathcal{P}_n^{\mathrm{N}^3\mathrm{MHV}} = \sum_{2\leq a_1,b_1\leq n-1} R_{n;a_1b_1}\Bigg[& \sum_{a_1+1\leq a_2,b_2\leq b_1} R^{0;a_1b_1}_{n;b_1a_1;a_2b_2}\Bigg(\sum_{a_2+1\leq a_3,b_3\leq b_2} R^{0;b_1a_1a_2b_2}_{n;b_1a_1;b_2a_2;a_3b_3} \\
& \qquad\qquad + \sum_{b_2\leq a_3,b_3\leq b_1} R^{b_1a_1a_2b_2;a_1b_1}_{n;b_1a_1;a_3b_3} + \sum_{b_1\leq a_3,b_3\leq n-1} R^{a_1b_1;0}_{n;a_3b_3}\Bigg) \\
& + \sum_{b_1\leq a_2,b_2\leq n-1} R^{a_1b_1;0}_{n;a_2b_2}\Bigg(\sum_{a_2+1\leq a_3,b_3\leq b_2} R^{0;a_2b_2}_{n;b_2a_2;a_3b_3} + \sum_{b_2\leq a_3,b_3\leq n-1} R^{a_2b_2;0}_{n;a_3b_3}\Bigg)\Bigg]\,.
\end{aligned}
\qquad (5.3)
$$

The reason we group the terms in this way is that it fits very naturally, together with formulae (3.17) and (4.9) for the NMHV and NNMHV cases, into a general pattern which we will now describe.

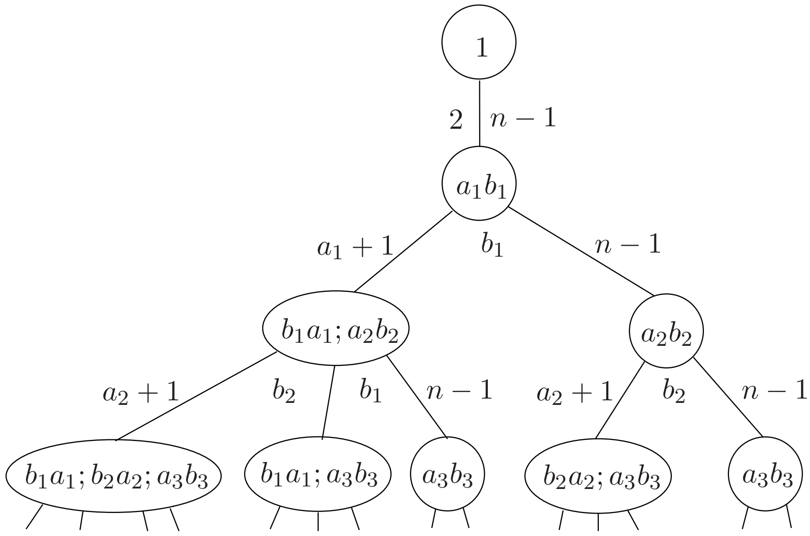

Figure 4. Graphical representation of the formula for tree-level amplitudes in $\mathcal{N} = 4$ SYM.

In the remainder of this section we will prove a general formula for all tree-level amplitudes in $\mathcal{N} = 4$ super Yang-Mills. First we must state the result. In order to do so we need to introduce a diagrammatic way of organising the general formula. Then we will go on to prove the formula by induction.

We illustrate the full n-point super-amplitude in figure 4 as a tree diagram, where the vertices correspond to the different R-invariants which appear. We consider a rooted tree, with the top vertex (the root) denoted by 1. The root has a single descendant vertex with labels a_1, b_1 and the tree is completed by passing from each vertex to a number of descendant vertices, as described in figure 5. We will enumerate the rows by $0, 1, 2, 3, \ldots$ with 0 corresponding to the root. For an n-point super-amplitude (with $n \geq 4$) only the rows up to row $n - 4$ in the tree will contribute to the amplitude.[2] The rule for completing the tree as given in figure 5 can be easily seen to imply that the number of vertices in row p is the Catalan number $C(p) = (2p)!/(p!(p+1)!)$.

Each vertex in the tree corresponds to an R-invariant with first label n and the remaining labels corresponding to those written in the vertex. For example, the first descendant vertex corresponds to the invariant $R_{n;a_1b_1}$ which we already saw appearing from the NMHV level. The next descendant vertices correspond to $R_{n;b_1a_1;a_2b_2}$ (which appears for the first time at NNMHV level) and $R_{n;a_2b_2}$, etc.

We consider vertical paths in the tree, starting from the root vertex at the top of figure 4. To each path we associate the product of the R-invariants (vertices) visited by the path, with a nested summation over all labels. The last pair of labels in a given vertex correspond to the ones which are summed first, i.e. the ones of the inner-most sum. In row p they are denoted by a_p, b_p. We always take the convention that $a_p + 2 \leq b_p$, which is needed for the corresponding R-invariant to be well-defined.

The lower and upper limits for the summation over the pair of labels a_p, b_p are noted

[2] The three-point MHV amplitude is a special case where only the root vertex contributes.

to the left and right of the line above each vertex in row p. For example, the labels a_1 and b_1 of $R_{n;a_1,b_1}$, associated to the first descendant vertex, are to be summed over the region $2 \leq a_1, b_1 \leq n-1$, as always with the convention that $a_1 + 2 \leq b_1$. The labels a_2 and b_2 on the R-invariants associated to the next descendant vertices are summed over the region $a_1 + 1 \leq a_2, b_2 \leq b_1$ for the vertex on the left, and the region $b_1 \leq a_2, b_2 \leq n-1$ for the vertex on the right, in both cases with the condition $a_2 + 2 \leq b_2$.

As we have seen already in the case of the NNMHV amplitudes, sometimes the R-invariants need to be modified when the summation labels reach their limiting lower or upper values. We deal with this by writing superscripts on the corresponding R-invariants, as we described in equations (4.5), (4.6), (4.7). We will illustrate how to obtain the superscripts on each R-invariant by referring to a general cluster of vertices in row p with a common parent vertex in row $p-1$, as shown in figure 5. Firstly, the left superscript of the left-most vertex in the cluster and the right superscript of the right-most vertex are both 0, i.e. they indicate no replacements at these boundaries. Then for the rest, the left superscript associated to a given vertex coincides with the right superscript associated to the vertex to its left. Therefore we need only specify the right superscripts. These are given by taking the labels in the vertex, deleting the final pair a_p, b_p and then reversing the order of the last pair which remain. For example, the vertex second from the left in figure 5 corresponds to the following sum,

$$\sum_{b_{p-1} \leq a_p b_p \leq v_r} R^{v_1 u_1 \ldots v_r u_r a_{p-1} b_{p-1}; v_1 u_1 \ldots v_{r-1} u_{r-1} u_r v_r}_{n; v_1 u_1; \ldots; v_r u_r; a_p b_p} \,. \tag{5.4}$$

The right superscript on the R-invariant is determined by taking the labels in the vertex, $v_1 u_1, \ldots, v_r u_r, a_p b_p$, deleting the final pair $a_p b_p$, and then reversing the order of the final two which remain, namely $v_r u_r$. The left superscript coincides with the right superscript of the vertex to its left in figure 5, and so can be obtained by performing the same operation on the labels of that vertex.

The formula for the full super-amplitude $\mathcal{A}_n = \mathcal{A}_n^{\mathrm{MHV}} \mathcal{P}_n$ is given by the sum over all vertical paths of any length, starting from the root,

$$\mathcal{P}_n = \sum \text{vertical paths in figure 4.} \tag{5.5}$$

Let us now see how the formula (5.5) works for the first few cases. Firstly there is one path of length zero, where we start at the root (row zero) and do not go anywhere. The value of this path is simply 1 and it corresponds to the MHV amplitudes,

$$\mathcal{P}_n^{\mathrm{MHV}} = 1 \,. \tag{5.6}$$

There is one path of length one, where we start at the root and go one step to its unique descendant. This path gives us 1 from the root, multiplied by $R_{n;a_1 b_1}$ from the descendant vertex, summed over a_1, b_1 with lower limit 2 and upper limit $n-1$. There are no boundary replacements in the sum since there is only one R-invariant in the relevant cluster and so both its left and right superscripts are 0. So we obtain for the NMHV amplitudes,

$$\mathcal{P}_n^{\mathrm{NMHV}} = \sum_{2 \leq a_1, b_1 \leq n-1} R^{0;0}_{n;a_1 b_1} = \sum_{2 \leq a_1 b_1 \leq n-1} R_{n;a_1,b_1} \,, \tag{5.7}$$

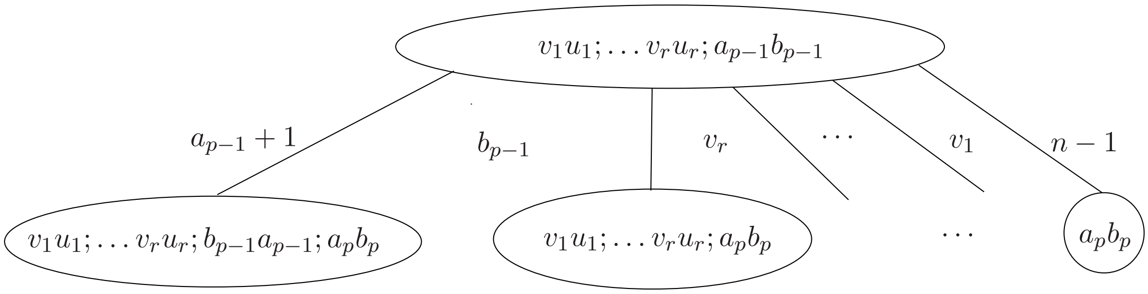

Figure 5. The rule for going from line $p-1$ to line p (for $p > 1$) in figure 4. For every vertex in line $p-1$ of the form given at the top of the diagram, there are $r+2$ vertices in the lower line (line p). The labels in these vertices start with $v_1u_1;\ldots v_ru_r;b_{p-1}a_{p-1};a_pb_p$ and they get sequentially shorter, with each step to the right removing the pair of labels adjacent to the last pair a_p, b_p until only the last pair is left. The summation limits between each line are also derived from the labels of the vertex above. The right superscripts associated to each vertex are obtained by deleting the final pair of labels a_pb_p and reversing the order the last pair which remain. The left superscript of a given vertex coincides with the right superscript of the vertex to its left.

which agrees with eq. (3.17).

There are two paths of length two. The first corresponds to descending from the root by one step and then descending once more to the left in figure 4. For this path we obtain 1 multiplied by $R_{n;a_1b_1}$ multiplied by $R_{n;b_1a_1;a_2b_2}$ with the limits for the outer sum over a_1, b_1 being the same as for the NMHV case above, while the inner sum, which is over a_2, b_2, has lower limit a_1+1 and upper limit b_1. The second path of length two corresponds to descending to the right instead of to the left. Doing so we obtain the product $1 \times R_{n;a_1b_1} \times R_{n;a_2b_2}$ with summation limits in the outer sum as before and in the inner sum being $b_1 \leq a_2, b_2 \leq n-1$.

The superscripts on the factors $R_{n;a_1b_1}$ are trivial as we just saw when looking at paths of length one. To obtain the superscripts on the other R-invariants, we recall that the left superscript of the left-most vertex in row 2 of figure 4 (corresponding to $R_{n;b_1a_1;a_2b_2}$) and also the right superscript of the right-most vertex (corresponding to $R_{n;a_2b_2}$) are 0. There is one non-trivial right superscript, that of $R_{n;b_1a_1;a_2b_2}$. It is obtained by deleting the final pair of indices a_2b_2 and reversing the order of the last pair which remains (which in this case is the pair b_1a_1). Thus we obtain $R^{0;a_1b_1}_{n;b_1a_1;a_2b_2}$. The left superscript of the other invariant is the same and so we obtain $R^{a_1b_1;0}_{n;a_2b_2}$.

Adding the two paths we obtain for the NNMHV amplitudes

$$\mathcal{P}_n^{\mathrm{NNMHV}} = \sum_{2\leq a_1,b_1\leq n-1} R^{0;0}_{n;a_1b_1}\left(\sum_{a_1+1\leq a_2,b_2\leq b_1} R^{0;a_1b_1}_{n;b_1a_1;a_2b_2} + \sum_{b_1\leq a_2,b_2\leq n-1} R^{a_1b_1;0}_{n;a_2b_2}\right), \tag{5.8}$$

which agrees with eq. (4.9).

Continuing, we find five paths of length three. Applying the rules for writing the sums over R-invariants and specifying their superscripts we find they correspond precisely to the five terms in the expression (5.3) for the N^{3}MHV amplitudes. Generically, since the number of vertices in row p of the tree in figure 4 is the Catalan number $C(p)$, we find $C(p)$

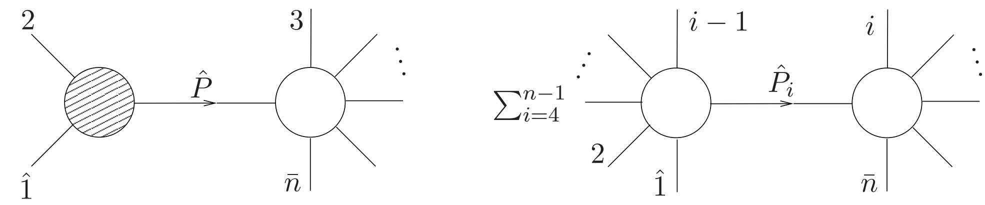

Figure 6. The two contributions to the r.h.s. of the supersymmetric recursion relation for the full super-amplitudes. We call the first term the linear term and the second term the quadratic term. As before $\hat{1}$ means that λ_1 is shifted, and $\bar{n}$ means that $\tilde{\lambda}_n$ is shifted.

terms in the expression for the N^{p}MHV amplitudes. Finally, by considering the sum of all vertical paths of any length, starting from the root, we obtain the sum of all amplitudes,

$$\mathcal{P}_n = \mathcal{P}_n^{\rm MHV} + \mathcal{P}_n^{\rm NMHV} + \mathcal{P}_n^{\rm NNMHV} + \cdots + \mathcal{P}_n^{\overline{\rm MHV}}. \tag{5.9}$$

The sum terminates (as it should) because, for a given value of n, there is maximum number of possible nestings beyond which all sums collapse to zero. This means that only paths up to length $n-4$ contribute and the longest paths correspond to the $\overline{\rm MHV}$ amplitudes. This completes the statement of the result for all tree-level amplitudes.

We will now prove the validity of formula (5.5), i.e. that all tree amplitudes are indeed given by summing vertical paths in the tree diagram figure 4. As usual we will proceed by induction and assume that the formula is correct for $(n-1)$-point amplitudes. The recursion relation for the full superamplitude $\mathcal{A}_n$ is illustrated in figure 6. All vertices except the $\overline{\rm MHV}_3$ vertex are full super-amplitudes. Specifically the relation reads

$$\mathcal{A}_n = \int \frac{d^4P}{P^2} \int d\eta_P \mathcal{A}_3^{\overline{\rm MHV}}(z_P)\mathcal{A}_{n-1}(z_P) + \sum_{i=4}^{n-1} \int \frac{d^4P_i}{P_i^2} \int d\eta_{P_i} \mathcal{A}_i(z_{P_i})\mathcal{A}_{n-i+2}(z_{P_i}). \tag{5.10}$$

We will call the first term on the r.h.s. side the linear term because it is linear in the full super-amplitude $\mathcal{A}$. Similarly we call the second term the quadratic term because there are two factors of $\mathcal{A}$ for each term in the sum over i.

We will introduce $\mathcal{P}_n$ into (5.10) in the usual way, $\mathcal{A}_n = \mathcal{A}_n^{\rm MHV}\mathcal{P}_n$. As in the particular cases of NMHV and NNMHV amplitudes, it is useful to insert the subamplitudes in this expression in our favourite orientations. We will choose the same orientations that we chose in those cases, i.e. a left factor will depend on the ordered set $\{2, \ldots, -\hat{P}, \hat{1}\}$ and a right factor on the ordered set $\{\hat{P}, \ldots, \bar{n}\}$. With this choice the recursion relation (5.10) reads

$$\begin{aligned}\mathcal{A}_n^{\rm MHV}\mathcal{P}_n = &\int \frac{d^4P}{P^2} \int d\eta_P \mathcal{A}_3^{\overline{\rm MHV}}(z_P)\mathcal{A}_{n-1}^{\rm MHV}\mathcal{P}_{n-1}(\hat{P}, 3, \ldots, \bar{n}) \\ &+\sum_{i=4}^{n-1}\int \frac{d^4P_i}{P_i^2} \int d\eta_{P_i} \mathcal{A}_i^{\rm MHV}\mathcal{P}_i(2, \ldots, -\hat{P}_i, \hat{1})\mathcal{A}_{n-i+2}^{\rm MHV}\mathcal{P}_{n-i+2}(\hat{P}_i, i, \ldots, \bar{n}).\end{aligned} \tag{5.11}$$

The reason for making this particular choice of orientations for the subamplitudes is the same as in the NMHV and NNMHV cases; the $\mathcal{P}$ factors are all inert under the Grassmann

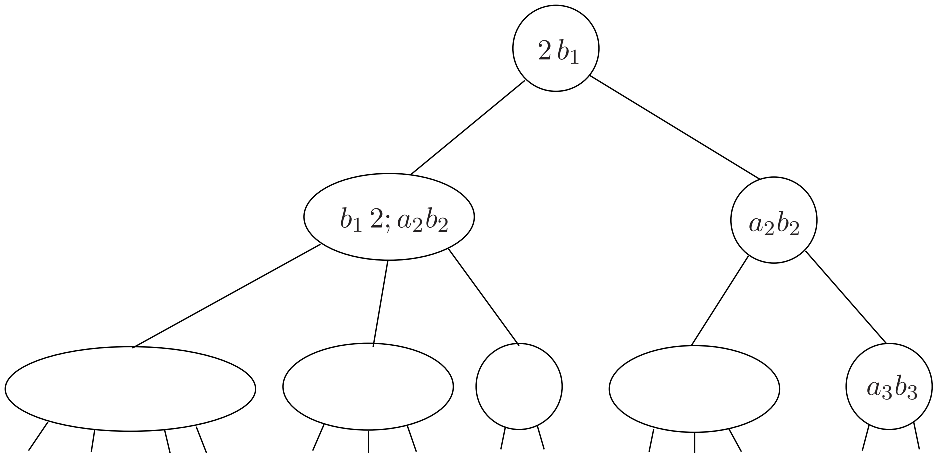

Figure 7. Graphical representation of the formula for the contributions missing from the linear term in the recursion relation. The variable b_1 is understood to be summed over the range $4 \le b_1 \le n-1$.

integral. This can be seen by looking at the η-dependence of the R-invariants appearing in the $\mathcal{P}$ factors, defined by the sum over paths in the tree diagram figure 4. The outer most R-invariant in each $\mathcal{P}$ factor is the same as in $\mathcal{P}^{\rm NMHV}$ (which we have already seen is inert with this choice of orientation) and the other R-invariants have at least as restrictive a range of η-dependence. This is just as we saw in the the case of the NNMHV amplitudes.

Once we have seen that the $\mathcal{P}$-factors are all inert, the Grassmann integrals in (5.11) are simple to do. The integration in the first term is the same as in the terms we called A in the NMHV and NNMHV cases, it provides the usual $\mathcal{A}_n^{\rm MHV}$ factor and leaves the $\mathcal{P}$-factor unchanged. In the second term, the Grassmann integration is the same as in the term we called B in the NMHV case or those we called B_1 and B_2 in the NNMHV case. We obtain a factor of $R_{n;2i}$ for each i as well as a factor of $\mathcal{A}_n^{\rm MHV}$. Thus we have

$$\mathcal{P}_n = \mathcal{P}_{n-1}(\hat{P}, 3, \ldots, \bar{n}) + \sum_{i=4}^{n-1} R_{n;2,i} \mathcal{P}_i(2, \ldots, -\hat{P}_i, \hat{1}) \mathcal{P}_{n-i+2}(\hat{P}_i, i, \ldots, \bar{n}). \tag{5.12}$$

As we saw already in the NMHV and NNMHV cases, the spinors $\langle \hat{P}|$ appearing in the R-invariants in the first term on the r.h.s. (the linear term) can be replaced by $\langle 2|$ due to the three-point kinematics. This term then gives an expression which is almost identical to the sum over paths in figure 4, except that the lower limit of the outermost sum is 3 and not 2. Thus to prove the inductive step we need to show that the second term in (5.12) (the quadratic term) gives the missing contributions, i.e. those paths of length one or greater where the outermost lower summation variable is fixed to be 2.

In fact the contributions we are looking for can also be represented diagrammatically as a sum over paths in a tree diagram very similar to the tree in figure 4. The tree representing the missing paths differs from figure 4 in that the root vertex is missing and the label a_1 is fixed to the value 2. We must remember that the label b_1 is still summed over the range $4 \le b_1 \le n-1$. We give the relevant tree diagram in figure 7.

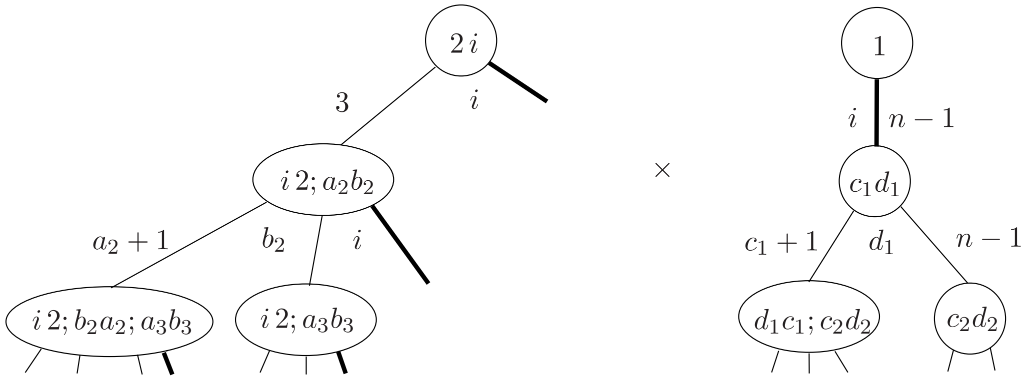

Figure 8. Graphical representation of the quadratic term $R_{n;2\,i}\mathcal{P}_i(2,\ldots,-\hat{P}_i,\hat{1})$ $\mathcal{P}_{n-i+2}(\hat{P}_i,i,\ldots,\bar{n})$ in equation (5.12). The left tree corresponds to the first two factors $R_{n;2\,i}\mathcal{P}_i(2,\ldots,-\hat{P}_i,\hat{1})$, while the right tree corresponds to the final factor $\mathcal{P}_{n-i+2}(\hat{P}_i,i,\ldots,\bar{n})$. As indicated in the text, after summing over i the first tree is almost what is needed to complete the linear term to $\mathcal{P}_n$. The missing pieces come from the right factor which can be adjoined to the left by inserting it everywhere there is a line drawn in bold so that these lines then all lead to a descendant vertex with labels c_2, d_2. Since the c and d labels are all dummy variables they can then be exchanged for the suitable a and b labels by a change of notation.

So let us examine the quadratic term in (5.12). We begin by looking at the summand. The first two factors $R_{n;2\,i}\mathcal{P}_i(2,\ldots,-\hat{P},\hat{1})$ taken together reproduce a sum over vertical paths in a tree very similar to the one in figure 4. The relevant tree diagram is shown in the left half of figure 8 (ignoring the solid lines for now). Let us describe the differences between this tree and the one of figure 4. Firstly, the root vertex corresponds to $R_{n;2\,i}$ instead of 1, so the first term in the sum over paths, 1, is absent. Secondly, the top R-invariant $R_{n;2\,i}$ has its labels fixed to be 2 and i. Thirdly, all descendant vertices have at least two pairs of labels due to the fact that the last leg of the argument of $\mathcal{P}_i(2,\ldots,-\hat{P},\hat{1})$ is $\hat{1}$ and not n. As we saw in equations (4.19) and (4.20) this results in the replacement $\langle n| \to \langle n|x_{ni}x_{i2}$ which induces extra labels on the R-invariants. Finally, the right-most vertex of each descendant cluster has two pairs of indices. Thus the right superscripts associated to these vertices are not 0, as was the case for the tree in figure 4. Instead these superscripts are all $2\,i$ which is obtained by deleting the final pair and reversing the order of the remaining pair $i\,2$.

Now let us consider the sum over vertical paths in the tree diagram we have just described. There is one path of length 0, corresponding to the contribution,

$$R_{n;2i}\,. \tag{5.13}$$

There is one path of length one which gives

$$R_{n;2i}\sum_{3\le a_2,b_2\le i} R^{0;2i}_{n;i2;a_2b_2}\,. \tag{5.14}$$

There are two paths of length two which give the following two contributions,

$$R_{n;2i}\sum_{3\le a_2,b_2\le i} R^{0;2i}_{n;i2;a_2b_2}\left[\sum_{a_2+1\le a_3,b_3} R^{0;i2a_2b_2}_{n;i2;b_2a_2;a_3b_3} + \sum_{b_2\le a_3,b_3\le i} R^{i2a_2b_2;2i}_{n;i2;a_3b_3}\right]. \tag{5.15}$$

Continuing, we have five paths of length three and so on. Since we consider the sum over paths, we have to add up all these terms.

Now we consider the third factor $\mathcal{P}_{n-i+2}(\hat{P}_i, i, \ldots, \bar{n})$ in the summand of the quadratic term in (5.12). This gives us the sum over paths in the tree shown in the right half of figure 8. This tree is again similar to the tree shown in figure 4. There are two differences between this tree and the one of figure 4. Firstly, the outermost lower summation limit is i and not 2. Secondly, since the first argument of $\mathcal{P}_{n-i+2}(\hat{P}_i, i, \ldots, \bar{n})$ is $\hat{P}_i$ and not $i-1$, there will be a non-trivial left superscript associated to the first descendant vertex. This is precisely the same effect that we saw in equations (4.13) and (4.14). The corresponding superscript is $2\,i$ so that the first descendant vertex corresponds to $R^{2i;0}_{n;c_1d_1}$.

Writing out the terms in the sum over paths in the tree corresponding to $\mathcal{P}_{n-i+2}(\hat{P}, i, \ldots, \bar{n})$ we find from paths of length 0,

$$1\,, \tag{5.16}$$

from paths of length one,

$$\sum_{i\le c_1,d_1\le n-1} R^{2i;0}_{n;c_1d_1}\,, \tag{5.17}$$

from paths of length two,

$$\sum_{i\le c_1,d_1\le n-1} R^{2i;0}_{n;c_1d_1}\Bigg[\sum_{c_1+1\le c_2,d_2\le d_1} R^{0;c_1d_1}_{n;d_1c_1;c_2d_2} + \sum_{d_1\le c_2,d_2\le n-1} R^{c_1d_1;0}_{n;c_2d_2}\Bigg], \tag{5.18}$$

and so on.

Thus the left half of figure 8 gave us the sum of (5.13), (5.14), (5.15) and longer paths. The right half of figure 4 gave us the sum of (5.16), (5.17), (5.18) and longer paths. If we consider the product of the expressions obtained from the two trees we see that it can be written,

$$\begin{aligned}
&R_{n;2i}\\
&+R_{n;2i}\Bigg[\sum_{3\le a_2,b_2\le i} R^{0;2i}_{n;i2;a_2b_2} + \sum_{i\le c_1,d_1\le n-1} R^{2i;0}_{n;c_1d_1}\Bigg]\\
&+R_{n;2i}\Bigg[\sum_{3\le a_2,b_2\le i} R^{0;2i}_{n;i2;a_2b_2}\Bigg[\sum_{a_2+1\le a_3,b_3} R^{0;i2a_2b_2}_{n;i2;b_2a_2;a_3b_3} + \sum_{b_2\le a_3,b_3\le i} R^{i2a_2b_2;2i}_{n;i2;a_3b_3} + \sum_{i\le c_1,d_1\le n-1} R^{2i;0}_{n;c_1d_1}\Bigg]\\
&\qquad + \sum_{i\le c_1,d_1\le n-1} R^{2i;0}_{n;c_1d_1}\Bigg[\sum_{c_1+1\le c_2,d_2\le d_1} R^{0;c_1d_1}_{n;d_1c_1;c_2d_2} + \sum_{d_1\le c_2,d_2\le n-1} R^{c_1d_1;0}_{n;c_2d_2}\Bigg]\Bigg] + \text{longer}\,.
\end{aligned} \tag{5.19}$$

Remembering that we need to sum over i in the quadratic term on the r.h.s. of (5.12), we find precisely the terms we are looking for. To make the identification completely explicit we can perform the changes of labels $c_1, d_1 \to a_2, b_2$ in the second line, $c_1, d_1 \to a_3, b_3$ in the third line and $c_1, d_1 \to a_2, b_2,\ c_2, d_2 \to a_3, b_3$ in the fourth line, and finally rename the summation variable i as b_1. This analysis can also be seen diagrammatically. If one glues the tree from the right half of figure 8 to that from the left half everywhere there is a line drawn in bold and performs the corresponding changes of labels, one obtains exactly the tree diagram of figure 7.

Thus finally we arrive at the fact that the sum of the linear term and quadratic term on the r.h.s. reproduces the sum over vertical paths in figure 4. This completes the inductive step of the proof. It remains to note that the sum over paths in figure 4 coincides with the first few amplitudes as we have seen by considering NMHV and NNMHV amplitudes. Therefore we conclude that formula (5.5) does indeed produce the full tree-level super-amplitude.

6 Symmetries of the amplitudes

Tree amplitudes in $\mathcal{N}=4$ SYM are expected to have many symmetries. First of all, $\mathcal{N}=4$ SYM is a superconformal field theory, so the amplitudes should exhibit this symmetry in their functional forms. The MHV super-amplitudes were shown to be annihilated by all generators of the conventional superconformal algebra in [6]. The amplitudes we have constructed in this paper are manifestly invariant under all generators of the conventional superconformal algebra[3] except for the superconformal symmetries $s, \bar{s}, k$.

In addition to the conventional superconformal symmetry, it was conjectured in [1] that the tree-level super-amplitudes should also exhibit *dual* superconformal symmetry. As far as tree-level super-amplitudes are concerned, the conjecture of [1] states that they should be covariant under dual conformal transformations K and the chiral superconformal transformations S, while they are invariant under $P, Q, \bar{Q}, \bar{S}$. They also have the obvious property that the dual dilatation weight and central charge are equal to n, the number of particles.

The generators of the two different realisations of the superconformal algebra are not all independent. As discussed in [1] the odd generator $\bar{q}$ coincides with $\bar{S}$, while $\bar{s}$ coincides with $\bar{Q}$. The same correspondence was observed in [2, 3] after performing a fermionic T-duality in the string sigma model. The explicit form of all generators is summarised in appendix B.

In [18] the dual conformal covariance of the tree-level super-amplitudes was verified recursively using the supersymmetric recursion relations. We can indeed see this symmetry in the explicit form of the solution we have presented. All quantities $R_{n;a_1b_1;\ldots;a_mb_m;st}$ are dual conformal invariants, as can be quickly verified by counting the conformal weights of the numerator and denominator. For tree-level amplitudes, this is sufficient to show dual superconformal covariance, as claimed in [18], since the conventional superconformal invariance $\bar{s}\mathcal{A}=0$ of the amplitude should be unbroken. In other words if we know that $\bar{s}\mathcal{A}=0$ then we have $\bar{Q}\mathcal{A}=0$, and together with covariance under dual inversions this is sufficient to derive all the expected properties under the full dual superconformal algebra. Further we remark that if all super-amplitudes obey $\bar{s}\mathcal{A}=0$ then they also obey $s\mathcal{A}=0$, since we could alternatively have performed the entire analysis in the anti-chiral $(\bar{\eta})$ representation for the gluon supermultiplet. Thus showing $\bar{s}$-invariance is sufficient to derive invariance under s and therefore under $k=\{s,\bar{s}\}$.

In general, showing the conventional superconformal invariance of the tree-level amplitudes is a non-trivial task (see e.g. [6]). Here we will explicitly show that expression (5.5)

[3]Following the conventions of [1] we will use lower case characters to denote the conventional superconformal generators and upper case ones for the dual superconformal generators.

does indeed obey this symmetry. As we have seen, the only property of the super-amplitude which remains to be explicitly verified is its behaviour under the $\bar{s}^A_{\dot\alpha}$ or $\bar{Q}^A_{\dot\alpha}$ supersymmetry. We recall the explicit form of $\bar{Q}^A_{\dot\alpha}$,

$$\overline{Q}^A_{\dot\alpha} = \sum_i [\theta_i^{\alpha A}\partial_{i\alpha\dot\alpha} + \eta_i^A \partial_{i\dot\alpha}] \,. \tag{6.1}$$

The invariance of the NMHV super-amplitude (3.17) was shown in [1]. It follows from the fact that[4]

$$\bar{Q}^A_{\dot\alpha}\, \delta^{(4)}(p)\, \delta^{(8)}(q)\, R_{n;a_1b_1} = 0 \,. \tag{6.2}$$

Following [1], we can simplify calculations such as (6.2) by noting that the super-amplitudes are invariant under $\bar{S}_{\dot\alpha A}$ and $Q_{\alpha A}$. Since translations, Lorentz rotations and the combination $D - C$ are also symmetries of the super-amplitudes, we have (see appendix B) $\left\{\bar{Q}^{\dot\alpha A}, \bar{S}_{\dot\beta B}\right\}\mathcal{A}_n = \left\{\bar{Q}^{\dot\alpha A}, Q_{\alpha B}\right\}\mathcal{A}_n = 0$. This allows us to compute the variation $\bar{Q}^A_{\dot\alpha}\mathcal{A}_n$ in a frame obtained by a combined shift using $\bar{S}_{\dot\alpha A}$ and $Q_{\alpha A}$. In particular, we can choose the shift parameters such that $\theta_{a_1} = \theta_{b_1} = 0$ [1].

Let us proceed with the NNMHV super-amplitude (4.9). We first consider terms in (4.9) which are not affected by boundary effects, i.e. $R_{n;a_1b_1}R_{n;a_2b_2}$ and $R_{n;a_1b_1}R_{n;b_1a_1;a_2b_2}$. From (6.2) we immediately see that the terms with $R_{n;a_1b_1}R_{n;a_2b_2}$ are invariant under $\bar{Q}^A_{\dot\alpha}$. Let us now consider the variation

$$\bar{Q}^A_{\dot\alpha}\, \delta^{(4)}(p)\, \delta^{(8)}(q)\, R_{n;a_1b_1} R_{n;b_1a_1;a_2b_2} = \delta^{(4)}(p)\, \delta^{(8)}(q)\, R_{n;a_1b_1}\, \bar{Q}^A_{\dot\alpha}\, R_{n;b_1a_1;a_2b_2} \,. \tag{6.3}$$

Following [1], we can choose a fixed frame in which $\theta_{a_2} = \theta_{b_2} = 0$. In this frame, (4.2) simplifies to

$$\Xi^A_{n;b_1a_1;a_2b_2} = x^2_{a_2b_2}\langle\xi\theta^A_{a_1}\rangle \tag{6.4}$$

and

$$R_{n;b_1a_1;a_2b_2} = \frac{1}{4!}\epsilon_{ABCD}\frac{\langle\xi\,\theta^A_{a_1}\rangle\langle\xi\,\theta^B_{a_1}\rangle\langle\xi\,\theta^C_{a_1}\rangle\langle\xi\,\theta^D_{a_1}\rangle}{\langle\xi\, I_1\rangle\langle\xi\, I_2\rangle\langle\xi\, I_3\rangle\langle\xi\, I_4\rangle}\,(x^2_{a_2b_2})^3\langle a_2\, a_2-1\rangle\langle b_2\, b_2-1\rangle \,. \tag{6.5}$$

Here

$$|I_1\rangle = x_{a_1a_2}x_{a_2b_2}|b_2\rangle,\, |I_2\rangle = x_{a_1a_2}x_{a_2b_2}|b_2-1\rangle,\, |I_3\rangle = x_{a_1b_2}x_{b_2a_2}|a_2\rangle,\, |I_4\rangle = x_{a_1b_2}x_{b_2a_2}|a_2-1\rangle \tag{6.6}$$

and

$$\langle\xi| = \langle n|x_{nb_1}x_{b_1a_1} \,. \tag{6.7}$$

Further, when computing the $\bar{Q}^{A\,\dot\alpha}$-variation of $R_{n;b_1a_1;a_2b_2}$ in (6.5) we can drop all terms in (6.1) except $\theta^A_{n\,\alpha}\partial^{\alpha\dot\alpha}_n + \theta^A_{a_1\,\alpha}\partial^{\alpha\dot\alpha}_{a_1} + \theta^A_{b_1\,\alpha}\partial^{\alpha\dot\alpha}_{b_1}$. The reason is that there is no explicit dependence on $\tilde\lambda$ in $R_{n;b_1a_1;a_2b_2}$, and that $\theta^A_{a_2\,\alpha}\partial^{\alpha\dot\alpha}_{a_2} = \theta^A_{b_2\,\alpha}\partial^{\alpha\dot\alpha}_{b_2} = 0$ in the fixed frame. Let $\tilde\lambda^{\dot\alpha}_J$ be an arbitrary projection. It can be easily seen that in the fixed frame, $[J\bar{Q}^E]$ acts trivially on I_i in (6.5), because e.g.

$$\langle\xi[J\bar{Q}^E]I_1\rangle = \langle\xi\,\theta^E_{a_1}\rangle[J|x_{a_2b_2}|b_2\rangle \tag{6.8}$$

[4] We omit the factor $\langle 12\rangle\ldots\langle n1\rangle$ in the denominator of the superamplitude since this is obviously invariant under the action of $\overline{Q}$.

is annihilated by the Grassmann delta function in the numerator of (6.5). Thus when acting with $[J\bar{Q}^E]$ on (6.5), only $\langle\xi|$ transforms. After using the cyclic identity for spinors we easily obtain

$$[J\bar{Q}^E]R_{n;b_1a_1;a_2b_2} = \frac{1}{4!}\epsilon_{ABCD}\frac{\chi^A\langle\xi\theta^B_{a_1}\rangle\langle\xi\theta^C_{a_1}\rangle\langle\xi\theta^D_{a_1}\rangle}{\langle\xi I_1\rangle\langle\xi I_2\rangle\langle\xi I_3\rangle\langle\xi I_4\rangle}\times\left[\langle n|x_{nb_1}x_{b_1a_1}|\theta^E_{a_1n}\rangle + \langle n|x_{na_1}x_{a_1b_1}|\theta^E_{b_1n}\rangle\right], \tag{6.9}$$

where the explicit expression for χ^A is inessential to our argument. From (6.9) we see that $R_{n;b_1a_1;a_2b_2}$ is *not* dual superconformally invariant. However, in (4.9), it always appears multiplied by the invariant $R_{n;a_1b_1}$. In this case, the Grassmann delta function in $R_{n;a_1b_1}$ makes the variation (6.9) vanish, and therefore $R_{n;a_1b_1}R_{n;b_1a_1;a_2b_2}$ is a dual superconformal invariant. The boundary terms in the sums behave in a similar way. The replacement spinors produce additional terms in the $\bar{Q}$ variation which are annihilated by the presence of the Grassmann factors.

We conclude that the NNMHV amplitudes are dual superconformally covariant. From the discussion here and in section 5 it is easy to see that this property is true for all tree-level amplitudes in $\mathcal{N}=4$ SYM. Indeed, one can repeat the argument above to 'longer' chains of invariants that appear in equation (5.5). Take for example $R_{n;a_1b_1}R_{n;b_1a_1;a_2b_2}R_{n;b_1a_1;b_2a_2;a_3b_3}$ from (5.2). After fixing a frame where $\theta_{a_3}=\theta_{b_3}=0$, we obtain an expression like (6.5) with a different $\langle\xi| = \langle n|x_{nb_1}x_{b_1a_1}x_{a_1b_2}x_{b_2a_2}|$. Because of the linearity of $[J\bar{Q}^E]$ the calculation of the variation of $R_{n;b_1a_1;b_2a_2;a_3b_3}$ is as above, except that now we obtain two contributions, one of which vanishes thanks to $R_{n;a_1b_1}$, and the other thanks to $R_{n;b_1a_1;a_2b_2}$. The crucial feature is that R's with many indices share all first indices of their 'predecessors'. This is the case by construction for all terms in (5.5).

Therefore we have shown explicitly that the formula (5.5) for all tree-level amplitudes in $\mathcal{N}=4$ SYM has all the expected properties under both conventional and dual superconformal symmetry.

7 Gluon scattering amplitudes from super-amplitudes

Here we wish to give some explanations on how gluon amplitudes can be extracted from our solutions and how this can be implemented, for example on a computer.

Let us first stress that any component amplitudes for arbitrary particle or helicity choice can be extracted from the super-amplitudes, see e.g. [1] for more explanations. Here we focus on the particularly simple case of gluon amplitudes.

According to (2.2), to each negative helicity gluon at position j is associated a factor of $(\eta_j)^4 = \eta^1_j\eta^2_j\eta^3_j\eta^4_j$, and to each positive helicity gluon simply a factor of 1. Going from a given super-amplitude to a gluon component amplitude therefore just amounts to extracting specific prefactors in the η-expansion of the super-amplitude. An elementary example is the relation (2.7) between the gluon MHV amplitude (2.1) and the super-amplitude (2.4).

A less trivial example is the split-helicity NMHV amplitude,

$$\mathcal{A}^{\text{NMHV}}_n = (\eta_{n-2})^4\,(\eta_{n-1})^4\,(\eta_n)^4\,A(1^+,\ldots,(n-3)^+,(n-2)^-,(n-1)^-,n^-) + \cdots\,, \tag{7.1}$$

We want to expand $\mathcal{A}_n^{\text{NMHV}}$ in η and recover the desired split-helicity gluon amplitude.[5] A simple way to achieve this is to observe that the relation between NMHV super-amplitude and the desired gluon component can be written as a Grassmann integral

$$A(1^+,\ldots,(n-3)^+,(n-2)^-,(n-1)^-,n^-) = \int d^4\eta_{n-2} \int d^4\eta_{n-1} \int d^4\eta_n\, \mathcal{A}_n^{\text{NMHV}}. \quad (7.2)$$

In this paper we have already encountered many such Grassmann integrals and seen that they are easy to do. We can always choose two arbitrary spinor projections of q_α^A to rewrite the $\delta^{(8)}(q_\alpha^A)$ of $\mathcal{A}_n^{\text{NMHV}}$ as

$$\delta^{(8)}(q_\alpha^A) = \langle n-1\, n\rangle^4\, \delta^{(4)}\left(\eta_{n-1}^A + \sum_{i=1}^{n-2} \frac{\langle in\rangle}{\langle n-1\, n\rangle}\eta_i^A\right) \delta^{(4)}\left(\eta_n^A + \sum_{i=1}^{n-2} \frac{\langle n-1\, i\rangle}{\langle n-1\, n\rangle}\eta_i^A\right). \quad (7.3)$$

This allows us to immediately carry out the $d^4\eta_n$ and $d^4\eta_{n-1}$ integrals in (7.2). The remaining terms in $\mathcal{A}_n^{\text{NMHV}}$ are unaffected by this since they can be written in the form (3.8) in which they are independent of η_{n-1} and η_n. Hence we obtain

$$A(1^+,\ldots,(n-3)^+,(n-2)^-,(n-1)^-,n^-) = \delta^{(4)}(p)\, \frac{\langle n-1\, n\rangle^4}{\langle 12\rangle \ldots \langle n1\rangle} \int d^4\eta_{n-2} \sum_{1<s,t<n} R_{n;s,t}\,, \quad (7.4)$$

where the $\Xi_{n;s,t}$ in $R_{n;s,t}$ are written in the form (3.8). A further simplification occurs because $R_{n;s,t}$ only depends on η_{n-2} if $t=n-1$, see (3.8). Carrying out the remaining Grassmann integration using the $\delta^{(4)}(\Xi_{n;s,n-1})$ in $R_{n;s,n-1}$ we obtain

$$\begin{aligned} &A(1^+,\ldots,(n-3)^+,(n-2)^-,(n-1)^-,n^-) = \\ &\quad -\frac{\delta^{(4)}(p)}{\langle 12\rangle \ldots \langle n-3\, n-2\rangle\langle n1\rangle} \sum_{s=2}^{n-3} \frac{\langle n-2|x_{n-1\,s}x_{sn}|n\rangle^3 \langle s\, s-1\rangle}{x_{s\,n-1}^2 x_{s\,n}^2 [n-1|x_{n-1,s}|s\rangle\, [n-1|x_{n-1\,s}|s-1\rangle}. \end{aligned} \quad (7.5)$$

This is in perfect agreement with formula (4.5) given in [34].

We can continue further and derive, for example, a formula for the split-helicity NN-MHV amplitudes. Just as in the NMHV case, we can write all invariants so that they do not depend on η_n or η_{n-1}. Then performing integrals with respect to these variables just produces a factor of $\langle n-1\, n\rangle^4$ from the $\delta^8(q)$ factor. The remaining integrals with respect to η_{n-2} and η_{n-3} give nothing from the second term in (4.9). From the first term in (4.9) we obtain two contributions, one where $b_1 = n-1$ and $b_2 = n-2$ and one where $b_1 = b_2 = n-1$. The final formula for the gluon amplitudes is

$$A(1^+,\ldots,(n-4)^+,(n-3)^-,(n-2)^-,(n-1)^-,n^-) = \delta^{(4)}(p)(S_1+S_2), \quad (7.6)$$

where the two terms are given by

$$S_1 = \frac{\langle n\, n-1\rangle\langle n-1\, n-2\rangle\langle n-2\, n-3\rangle}{\prod_{i=1}^n \langle i\, i+1\rangle} \sum_{a_1=2}^{n-5} \sum_{a_2=a_1+1}^{n-4} \frac{N_1}{D_1}, \quad (7.7)$$

[5]Note that a Grassmann delta function is simply defined as a product, $\delta^{(4)}(\chi^A) = 1/4!\epsilon_{ABCD}\chi^A\chi^B\chi^C\chi^D$.

$$S_2 = \frac{\langle n\, n-1\rangle\langle n-1\, n-2\rangle\langle n-2\, n-3\rangle^4}{\prod_{i=1}^n \langle i\, i+1\rangle} \sum_{a_1=2}^{n-4} \sum_{a_2=a_1+1}^{n-3} \frac{N_2}{D_2}. \tag{7.8}$$

Here the numerators and denominators of the summands are

$$N_1 =\langle a_1\, a_1-1\rangle\langle n|x_{na_1}x_{a_1 n-1}|n-2\rangle^3\langle a_2\, a_2-1\rangle[n-1|x_{n-1a_1}x_{a_1a_2}x_{a_2n-3}|n-3\rangle^3, \tag{7.9}$$

$$\begin{aligned} D_1 =&[n-1|x_{n-1a_1}|a_1\rangle[n-1|x_{n-1a_1}|a_1-1\rangle[n-1|x_{n-1a_1}x_{a_1a_2}x_{a_2n-2}|n-2\rangle \\ &\times [n-1|x_{n-1a_1}x_{a_1n-2}x_{n-2a_2}|a_2\rangle \\ &\times [n-1|x_{n-1a_1}x_{a_1n-2}x_{n-2a_2}|a_2-1\rangle x^2_{a_1n-1}x^2_{na_1}x^2_{a_2n-2}, \end{aligned} \tag{7.10}$$

$$N_2 =\langle a_1a_1-1\rangle\langle a_2a_2-1\rangle\langle n|x_{na_1}x_{a_1n-1}x_{n-1a_2}x_{a_2a_1}x_{a_1n-1}|n-1]^3, \tag{7.11}$$

$$\begin{aligned} D_2 =&[n-1|x_{n-1a_1}|a_1\rangle[n-1|x_{n-1a_1}|a_1-1\rangle[n-1|x_{n-1a_1}x_{a_1a_2}x_{a_2n-1}|n-2\rangle \\ &\times [n-1|x_{n-1a_2}|a_2\rangle[n-1|x_{n-1a_2}|a_2-1\rangle(x^2_{a_1n-1})^3x^2_{a_2n-1}x^2_{na_1}. \end{aligned} \tag{7.12}$$

It is simple to check analytically that this formula correctly reproduces the six-point $\overline{\text{MHV}}$ amplitude and the seven-point next-to-$\overline{\text{MHV}}$ amplitude. We have also checked numerically that it coincides with the six terms given in [34] for the eight-point NNMHV split-helicity gluon amplitude.

In more complicated situations one could for example first do some η integrations analytically (e.g. using the $\delta^{(8)}(q)$ which is present in all physical super-amplitudes because of supersymmetry), and then implement the remaining integrations/expansions on a computer. This can be easily programmed, keeping track of the overall sign (because the η's are anticommuting variables). The resulting spinor expressions can be evaluated numerically using available packages, see e.g. [35].

8 Conclusions

The main result of our paper is formula (5.5) for all tree-level amplitudes in $\mathcal{N} = 4$ SYM. The formula contains all amplitudes with arbitrary total helicity (MHV,NMHV,...,$\overline{\text{MHV}}$). It is given in terms of vertical paths of a particular rooted tree, shown in figure 4. This extends previous solutions of the BCF recursion relations which applied only to the closed subset of split-helicity gluon amplitudes [34]. Our solution is written in on-shell $\mathcal{N} = 4$ superspace. It is built from dual superconformal invariants and so it manifestly exhibits both conventional and dual superconformal symmetries.

Our expression contains as components all amplitudes for arbitrary external states and helicities. We explained in section 7 that gluon components are particularly simple to extract, since they can be obtained from the super-amplitudes by carrying out Grassmann integrations. A crucial simplifying feature is that (5.5) is built from sums over products of Grassmann delta functions, which can be used to perform the aforementioned integrations. We expect that it will be possible to obtain compact expressions for previously unknown gluon components following the example in section 7.

We expect our results to be relevant for $\mathcal{N} = 8$ supergravity as well, since tree-level amplitudes in the latter theory can be obtained from those in $\mathcal{N} = 4$ SYM through the KLT relations [36]. Furthermore the methods employed here could also be directly applied to solving recursion relations for supergravity tree-level amplitudes [37]. It would also be interesting to see if our formula could shed light on the relation among tree-level amplitudes described in [38].

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A Collinear limit of the super-amplitudes

Here we check that our amplitudes have the correct collinear limit as two particles become almost collinear [39]. Consider two neighbouring particles at points a and $b = a+1$ that become collinear such that

$$p_a = zP\,, \qquad p_b = (1-z)P\,, \tag{A.1}$$

then an n-gluon tree amplitude is expected to behave as

$$A_n \xrightarrow{a||b} \sum_{\lambda=\pm} \mathrm{Split}^{\mathrm{tree}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})\, A_n(\ldots, (a+b)^{\lambda}, \ldots)\,, \tag{A.2}$$

where $\mathrm{Split}^{\mathrm{tree}}_{-\lambda}$ are certain helicity-dependent splitting functions, see [39]. The non-vanishing splitting functions diverge as $1/\sqrt{s_{ab}}$ in the collinear limit $s_{ab} = (p_a + p_b)^2 \to 0$. In the collinear limit, the spinors corresponding to the momenta p_a and p_b become

$$\lambda_a \to \sqrt{z}\lambda_P\,, \qquad \tilde\lambda_a \to \sqrt{z}\tilde\lambda_P\,, \qquad \lambda_b \to \sqrt{1-z}\lambda_P\,, \qquad \tilde\lambda_b \to \sqrt{1-z}\tilde\lambda_P\,. \tag{A.3}$$

In the supersymmetric case, to be consistent with (A.3) we also define

$$\eta_a \to \sqrt{z}\eta_P\,, \qquad \eta_b \to \sqrt{1-z}\eta_P\,. \tag{A.4}$$

By inspecting the collinear limit for the MHV super-amplitudes (2.4), we expect the following collinear limit for super-amplitudes at tree level,

$$\mathcal{A}_n(\ldots, a, b, \ldots) \xrightarrow{a||b} \frac{1}{\sqrt{z(1-z)}\langle ab\rangle}\mathcal{A}_{n-1}(\ldots, P, \ldots)\,. \tag{A.5}$$

Let us see if relation (A.5) holds for the NMHV amplitudes (2.8) as well. We need to analyse the behaviour of the invariants $R_{n;s,t}$ in the limit. Because of cyclic symmetry of the super-amplitude, we can consider the $a = n-1, b = n$ without loss of generality. This is advantageous because then the invariants $R_{n;s,t}$ are affected by the collinear limit only through $\lambda_n = \sqrt{1-z}\lambda_P$. Looking at (3.5) we see that

$$R_{n;s,t} \xrightarrow{n-1||n} R_{P;s,t}\,. \tag{A.6}$$

We also observe that

$$R_{n;s,n-1} \xrightarrow{n-1||n} R_{P;s,n-1} \propto \langle n-1\, n\rangle^2 \to 0\,. \tag{A.7}$$

Using (A.6) and (A.7) on (2.8) we see that indeed

$$\mathcal{A}_n^{\rm NMHV}(1,\ldots,n-1,n) \stackrel{n-1||n}{\longrightarrow} \frac{1}{\sqrt{z(1-z)}\langle n-1\, n\rangle}\mathcal{A}_{n-1}^{\rm NMHV}(1,\ldots,n-2,P)\,. \qquad (A.8)$$

Going to NNMHV amplitudes (4.8), (4.9), we see that the behaviour of the 'longer' invariants like $R_{n;u,v}R_{n;v,u;s,t}$ under the collinear limit where particles $n-1$ and n become collinear is completely analogous to the NMHV case, they turn into $R_{P;u,v}R_{P;v,u;s,t}$. It is then obvious that (4.8), (4.9) obeys the collinear limit (A.5). This observation can be immediately generalised to arbitrary non-MHV amplitudes. The crucial feature is that all invariants share the same first label n, which is simply replaced by P in the collinear limit.

Finally we remark that the divergent prefactor in (A.5) originates entirely from the MHV prefactor $\mathcal{A}_n^{\rm MHV}$, and that $\mathcal{P}_n$ (defined in (2.5)) has a finite collinear limit.

B Conventional and dual superconformal generators

In this appendix we give the conventional and dual representations of the superconformal algebra. We begin by listing the commutation relations of the algebra $u(2,2|4)$. The Lorentz generators $\mathbb{M}_{\alpha\beta}$, $\overline{\mathbb{M}}_{\dot\alpha\dot\beta}$ and the $su(4)$ generators $\mathbb{R}^A{}_B$ act canonically on the remaining generators carrying Lorentz or $su(4)$ indices. The dilatation $\mathbb{D}$ and hypercharge $\mathbb{B}$ act via

$$[\mathbb{D},\mathbb{J}] = \dim(\mathbb{J}), \qquad [\mathbb{B},\mathbb{J}] = {\rm hyp}(\mathbb{J}). \qquad (B.1)$$

The non-zero dimensions and hypercharges of the various generators are

$$\begin{aligned} &\dim(\mathbb{P})=1, && \dim(\mathbb{Q})=\dim(\overline{\mathbb{Q}})=\frac{1}{2}, && \dim(\mathbb{S})=\dim(\overline{\mathbb{S}})=-\frac{1}{2} \\ &\dim(\mathbb{K})=-1, && {\rm hyp}(\mathbb{Q})={\rm hyp}(\overline{\mathbb{S}})=\frac{1}{2}, && {\rm hyp}(\overline{\mathbb{Q}})={\rm hyp}(\mathbb{S})=-\frac{1}{2}. \end{aligned} \qquad (B.2)$$

The remaining non-trivial commutation relations are,

$$\begin{aligned} \{\mathbb{Q}_{\alpha A},\overline{\mathbb{Q}}^B_{\dot\alpha}\} &= \delta^B_A \mathbb{P}_{\alpha\dot\alpha}, & \{\mathbb{S}^A_\alpha,\overline{\mathbb{S}}_{\dot\alpha B}\} &= \delta^A_B\mathbb{K}_{\alpha\dot\alpha}, \\ [\mathbb{P}_{\alpha\dot\alpha},\mathbb{S}^{\beta A}] &= \delta^\beta_\alpha \overline{\mathbb{Q}}^A_{\dot\alpha}, & [\mathbb{K}_{\alpha\dot\alpha},\mathbb{Q}^\beta_A] &= \delta^\beta_\alpha\overline{\mathbb{S}}_{\dot\alpha A}, \\ [\mathbb{P}_{\alpha\dot\alpha},\overline{\mathbb{S}}^{\dot\beta}_A] &= \delta^{\dot\beta}_{\dot\alpha}\mathbb{Q}_{\alpha A}, & [\mathbb{K}_{\alpha\dot\alpha},\overline{\mathbb{Q}}^{\dot\beta A}] &= \delta^{\dot\beta}_{\dot\alpha}\mathbb{S}^A_\alpha, \\ [\mathbb{K}_{\alpha\dot\alpha},\mathbb{P}^{\beta\dot\beta}] &= \delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha}\mathbb{D} + \mathbb{M}_\alpha{}^\beta\delta^{\dot\beta}_{\dot\alpha} + \overline{\mathbb{M}}_{\dot\alpha}{}^{\dot\beta}\delta^\beta_\alpha, \\ \{\mathbb{Q}^\alpha_A,\mathbb{S}^B_\beta\} &= \mathbb{M}^\alpha{}_\beta\delta^B_A + \delta^\alpha_\beta\mathbb{R}^B{}_A + \frac{1}{2}\delta^\alpha_\beta\delta^B_A(\mathbb{D}+\mathbb{C}), \\ \{\overline{\mathbb{Q}}^{\dot\alpha A},\overline{\mathbb{S}}_{\dot\beta B}\} &= \overline{\mathbb{M}}^{\dot\alpha}{}_{\dot\beta}\delta^A_B - \delta^{\dot\alpha}_{\dot\beta}\mathbb{R}^A{}_B + \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\delta^A_B(\mathbb{D}-\mathbb{C}). \end{aligned} \qquad (B.3)$$

Note that in writing the algebra relations we are obliged to choose the $su(4)$ chirality of the odd generators. The relations above are valid directly for the dual superconformal generators. For the conventional realisation of the algebra, one should simply swap all $su(4)$ chiralities appearing in the commutation relations. We now give the generators in both the conventional and dual representations of the superconformal algebra. We will use the following shorthand notation:

$$\partial_{i\alpha\dot\alpha} = \frac{\partial}{\partial x_i^{\alpha\dot\alpha}}, \qquad \partial_{i\alpha A} = \frac{\partial}{\partial\theta_i^{\alpha A}}, \qquad \partial_{i\alpha} = \frac{\partial}{\partial\lambda_i^\alpha}\,, \qquad \partial_{i\dot\alpha} = \frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}\,, \qquad \partial_{iA} = \frac{\partial}{\partial\eta_i^A}\,. \quad (B.4)$$

We first give the generators of the conventional superconformal symmetry, using lower case characters to distinguish these generators from the dual superconformal generators which follow afterwards.

$$
\begin{aligned}
p^{\dot\alpha\alpha} &= \sum_i \tilde\lambda_i^{\dot\alpha}\lambda_i^{\alpha}, & k_{\alpha\dot\alpha} &= \sum_i \partial_{i\alpha}\partial_{i\dot\alpha},\\
\overline{m}_{\dot\alpha\dot\beta} &= \sum_i \tilde\lambda_{i(\dot\alpha}\partial_{i\dot\beta)}, & m_{\alpha\beta} &= \sum_i \lambda_{i(\alpha}\partial_{i\beta)},\\
d &= \sum_i \left[\frac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} + \frac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} + 1\right], & r^A{}_B &= \sum_i \left[-\eta_i^A\partial_{iB} + \frac{1}{4}\delta^A_B\eta_i^C\partial_{iC}\right],\\
q^{\alpha A} &= \sum_i \lambda_i^{\alpha}\eta_i^A, & \bar q^{\dot\alpha}_A &= \sum_i \tilde\lambda_i^{\dot\alpha}\partial_{iA},\\
s_{\alpha A} &= \sum_i \partial_{i\alpha}\partial_{iA}, & \bar s^A_{\dot\alpha} &= \sum_i \eta_i^A\partial_{i\dot\alpha}.\\
c &= \sum_i \left[1 + \frac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} - \frac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} - \frac{1}{2}\eta_i^A\partial_{iA}\right] & &
\end{aligned}
\tag{B.5}
$$

We can construct the generators of dual superconformal transformations by starting with the standard chiral representation and extending the generators so that they commute with the constraints,

$$
(x_i - x_{i+1})_{\alpha\dot\alpha} - \lambda_{i\,\alpha}\,\tilde\lambda_{i\,\dot\alpha} = 0, \qquad (\theta_i - \theta_{i+1})^A_{\alpha} - \lambda_{i\alpha}\eta_i^A = 0. \tag{B.6}
$$

By construction they preserve the surface defined by these constraints, which is where the amplitude has support. The generators are

$$
P_{\alpha\dot\alpha} = \sum_i \partial_{i\alpha\dot\alpha}, \tag{B.7}
$$

$$
Q_{\alpha A} = \sum_i \partial_{i\alpha A}, \tag{B.8}
$$

$$
\overline{Q}^A_{\dot\alpha} = \sum_i [\theta_i^{\alpha A}\partial_{i\alpha\dot\alpha} + \eta_i^A\partial_{i\dot\alpha}], \tag{B.9}
$$

$$
M_{\alpha\beta} = \sum_i [x_{i(\alpha}{}^{\dot\alpha}\partial_{i\beta)\dot\alpha} + \theta^A_{i(\alpha}\partial_{i\beta)A} + \lambda_{i(\alpha}\partial_{i\beta)}], \tag{B.10}
$$

$$
\overline{M}_{\dot\alpha\dot\beta} = \sum_i [x_{i(\dot\alpha}{}^{\alpha}\partial_{i\dot\beta)\alpha} + \tilde\lambda_{i(\dot\alpha}\partial_{i\dot\beta)}], \tag{B.11}
$$

$$
R^A{}_B = \sum_i [\theta_i^{\alpha A}\partial_{i\alpha B} + \eta_i^A\partial_{iB} - \frac{1}{4}\delta^A_B\theta_i^{\alpha C}\partial_{i\alpha C} - \frac{1}{4}\delta^A_B\eta_i^C\partial_{iC}], \tag{B.12}
$$

$$
D = \sum_i [-x_i^{\dot\alpha\alpha}\partial_{i\alpha\dot\alpha} - \frac{1}{2}\theta_i^{\alpha A}\partial_{i\alpha A} - \frac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} - \frac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha}], \tag{B.13}
$$

$$
C = \sum_i [-\frac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} + \frac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} + \frac{1}{2}\eta_i^A\partial_{iA}], \tag{B.14}
$$

$$
S^A_{\alpha} = \sum_i [-\theta^B_{i\alpha}\theta_i^{\beta A}\partial_{i\beta B} + x_{i\alpha}{}^{\dot\beta}\theta_i^{\beta A}\partial_{\beta\dot\beta} + \lambda_{i\alpha}\theta_i^{\gamma A}\partial_{i\gamma} + x_{i+1\,\alpha}{}^{\dot\beta}\eta_i^A\partial_{i\dot\beta} - \theta^B_{i+1\,\alpha}\eta_i^A\partial_{iB}], \tag{B.15}
$$

$$\overline{S}_{\dot\alpha A} = \sum_i [x_{i\dot\alpha}{}^{\beta}\partial_{i\beta A} + \tilde\lambda_{i\dot\alpha}\partial_{iA}], \tag{B.16}$$

$$K_{\alpha\dot\alpha} = \sum_i [x_{i\alpha}{}^{\dot\beta} x_{i\dot\alpha}{}^{\beta}\partial_{i\beta\dot\beta} + x_{i\dot\alpha}{}^{\beta}\theta^{B}_{i\alpha}\partial_{i\beta B} + x_{i\dot\alpha}{}^{\beta}\lambda_{i\alpha}\partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot\beta}\tilde\lambda_{i\dot\alpha}\partial_{i\dot\beta} + \tilde\lambda_{i\dot\alpha}\theta^{B}_{i+1\,\alpha}\partial_{iB}]. \tag{B.17}$$

We also have the hypercharge B,

$$B = \sum_i [-\frac{1}{2}\theta_i^{\alpha A}\partial_{i\alpha A} - \frac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} + \frac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha}] \tag{B.18}$$

Note that if we restrict the dual generators $\bar{Q}, \bar{S}$ to the on-shell superspace they become identical to the conventional generators $\bar{s}, \bar{q}$.

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